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Calculus of Variations

Monge–Ampère equation on exterior domains

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Abstract We consider the Monge–Ampère equation $\det(D^2u) = f$ where f is a positive function in $\mathbb{R}^n$ and $f = 1 + O(|x|^{-\beta})$ for some $\beta > 2$ at infinity. If the equation is globally defined on $\mathbb{R}^n$ we classify the asymptotic behavior of solutions at infinity. If the equation is defined outside a convex bounded set we solve the corresponding exterior Dirichlet problem. Finally we prove for $n \geq 3$ the existence of global solutions with prescribed asymptotic behavior at infinity. The assumption $\beta > 2$ is sharp for all the results in this article.

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1 Introduction

It is well known that Monge–Ampère equations are a class of important fully nonlinear equations profoundly related to many fields of analysis and geometry. In the past few decades many significant contributions have been made on various aspects of Monge–Ampère equations. In particular, the Dirichlet problem

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$$\begin{cases} det(D^2u) = f, & \text{in} \quad D, \\ u = \phi & \text{on} \quad \partial D \end{cases}$$

on a convex, bounded domain D is completely understood through the works of Aleksandrov [1], Bakelman [2], Nirenberg [4], Calabi [12], Pogorelov [30,32,33], Cheng and Yau [13], Caffarelli et al. [11], Caffarelli [7], Ivochkina [22–24], Krylov [27], Jian and Wang [25], Huang [21], Trudinger-Wang [36], Urbas [38],Savin [34,35], Philippis and Figalli [29] and the references therein. Corresponding to the traditional Dirichlet problem mentioned above, there is an exterior Dirichlet problem which seeks to solve the Monge–Ampère equation outside a convex set. More specifically, let D be a smooth, bounded and strictly convex subset of $\mathbb{R}^n$ and let $\phi \in C^2(\partial D)$, the exterior Dirichlet problem is to find u to verify

$$\begin{cases} det(D^2u) = f(x), \quad x \in \mathbb{R}^n \backslash \overline{D}, \\ u \in C^0(\mathbb{R}^n \backslash D) \text{ is a locally convex viscosity solution,} \\ u = \phi, \quad \text{on} \quad \partial D. \end{cases} \tag{1.1}$$

If $f \equiv 1$ and $n \geq 3$, Caffarelli and Li [9] proved that any solution u of (1.1) is very close to a parabola near infinity. They solved the exterior Dirichlet problem assuming that u equals ϕ on ∂D and has a prescribed asymptotic behavior at infinity. For $f \equiv 1$ and $n = 2$, Ferrer et al. [18,19] used a method of complex analysis to prove that any solution u of (1.1) is very close to a parabola plus a logarithmic function at infinity (see also Delanoë [16]). Recently the first two authors [3] solved the exterior Dirichlet problem for $f \equiv 1$ and $n = 2$. In the first part of this article we solve the exterior Dirichlet problem assuming that f is a perturbation of 1 near infinity:

$$\begin{aligned} (FA): & f \in C^0(\mathbb{R}^n), \;\; 0 < \inf_{\mathbb{R}^n} f \leq \sup_{\mathbb{R}^n} f < \infty. \\ & \exists m \geq 3, \;\text{ such that } D^m f \text{ exists outside a compact subset of } \mathbb{R}^n, \\ & \exists \beta > 2 \text{ such that } \lim_{|x|\to\infty} |x|^{\beta+k} |D^k(f(x) - 1)| < \infty, \;\; k = 0, 1, \ldots, m. \end{aligned}$$

Let $\mathbb{M}^{n\times n}$ be the set of the real valued, $n \times n$ matrices and

$$\mathcal{A} := \{A \in \mathbb{M}^{n\times n} : \quad A \text{ is symmetric, positive definite and } det(A) = 1\}.$$

Our first main theorem is

Theorem 1.1 *Let D be a strictly convex, smooth and bounded set, $\phi \in C^2(\partial D)$ and f satisfy (FA). If $n \geq 3$, then for any $b \in \mathbb{R}^n$, $A \in \mathcal{A}$, there exists $c_*(n, D, \phi, b, A, f)$ such that for any $c > c_*$, there exists a unique u to (1.1) that satisfies*

$$\limsup_{|x|\to\infty} |x|^{\min\{\beta,n\}-2+k} |D^k \left(u(x) - \left(\frac{1}{2}x'Ax + b \cdot x + c\right)\right)| < \infty \tag{1.2}$$

for $k = 0, \ldots, m+1$. If $n = 2$, then for any $b \in \mathbb{R}^2$, $A \in \mathcal{A}$, there exists $d^* \in \mathbb{R}$ depending only on A, b, ϕ, f, D such that for all $d > d^*$, there exists a unique u to (1.1) that satisfies

$$\limsup_{|x|\to\infty} |x|^{k+\sigma} \left| D^k \left(u(x) - \left(\frac{1}{2}x'Ax + b \cdot x + d \log \sqrt{x'Ax} + c_d\right)\right)\right| < \infty \tag{1.3}$$

for $k = 0, \ldots, m+1$ and $\sigma \in (0, \min\{\beta - 2, 2\})$. $c_d \in \mathbb{R}$ is uniquely determined by D, ϕ, d, f, A, b.

The Dirichlet problem on exterior domains is closely related to asymptotic behavior of solutions defined on entire $\mathbb{R}^n$. The classical theorem of Jörgens [26], Calabi [12] and Pogorelov [31] states that any convex classical solution of $det(D^2u) = 1$ on $\mathbb{R}^n$ must be a quadratic polynomial. See Cheng and Yau [14], Caffarelli [7] and Jost and Xin [17] for different proofs and extensions. Caffarelli and Li [9] extended this result by considering

$$det(D^2u) = f \quad \mathbb{R}^n \tag{1.4}$$

where f is a positive continuous function and is not equal to 1 only on a bounded set. They proved that for $n \geq 3$, the convex viscosity solution u is very close to quadratic polynomial at infinity and for $n = 2$, u is very close to a quadratic polynomial plus a logarithmic term asymptotically. In a subsequent work [10] Caffarelli and Li proved that if f is periodic, then u must be a perturbation of a quadratic function.

The second main result of the paper is to extend the Caffarelli and Li results on global solutions in [9]:

Theorem 1.2 *Let $u \in C^0(\mathbb{R}^n)$ be a convex viscosity solution to (1.4) where f satisfies (FA). If $n \geq 3$, then there exist $c \in \mathbb{R}, b \in \mathbb{R}^n$ and $A \in \mathcal{A}$ such that (1.2) holds. If $n = 2$ then there exist $c \in \mathbb{R}, b \in \mathbb{R}^2, A \in \mathcal{A}$ such that (1.3) holds for $d = \frac{1}{2\pi}\int_{\mathbb{R}^2}(f-1)$ and $\sigma \in (0, \min\{\beta - 2, 2\})$.*

Corollary 1.1 *Let D be a bounded, open and convex subset of $\mathbb{R}^n$ and let $u \in C^0(\mathbb{R}^n \setminus \bar{D})$ be a locally convex viscosity solution to*

$$det(D^2u) = f, \quad in \quad \mathbb{R}^n \setminus \bar{D} \tag{1.5}$$

where f satisfies (FA). Then for $n \geq 3$, there exist $c \in \mathbb{R}, b \in \mathbb{R}^n$ and $A \in \mathcal{A}$ such that (1.2) holds. For $n = 2$, there exist $A \in \mathcal{A}, b \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$ such that for $k = 0, \ldots, m+1$

$$\limsup_{|x|\to\infty} |x|^{k+\sigma} \left| D^k \left(u(x) - \left(\frac{1}{2} x' A x + b \cdot x + d \log \sqrt{x' A x} + c \right) \right) \right| < \infty$$

holds for $\sigma \in (0, \min\{\beta - 2, 2\})$.

As is well known the Monge–Ampère equation $det(D^2u) = f$ is closely related to the Minkowski problems, the Plateau type problems, mass transfer problems, and affine geometry, etc. In many of these applications f is not a constant. The readers may see the survey paper of Trudinger and Wang [37] for more description and applications. The importance of f not identical to 1 is also mentioned by Calabi in [12].

Next we consider the globally defined equation (1.4) and the existence of global solutions with prescribed asymptotic behavior at infinity.

Theorem 1.3 *Suppose f satisfies (FA). Then for any $A \in \mathcal{A}, b \in \mathbb{R}^n$ and $c \in \mathbb{R}$, if $n \geq 3$ there exists a unique convex viscosity solution u to (1.4) such that (1.2) holds.*

The following example shows that the decay rate assumption $\beta > 2$ in (FA) is sharp in all the theorems. Let f be a radial, smooth, positive function such that $f(r) \equiv 1$ for $r \in [0, 1]$ and $f(r) = 1 + r^{-2}$ for $r > 2$. Let

$$u(r) = n^{\frac{1}{n}} \int_0^r \left(\int_0^s t^{n-1} f(t) dt \right)^{\frac{1}{n}} ds, \quad r = |x|.$$

It is easy to check that $det(D^2u) = f$ in $\mathbb{R}^n$. Moreover for $n \geq 3$, $u(x) = \frac{1}{2}|x|^2 + O(\log|x|)$ at infinity. For $n = 2$, $u(x) = \frac{1}{2}|x|^2 + O((\log|x|)^2)$ at infinity.

Corresponding to the results in this paper we make the following two conjectures. First we think the analogue of Theorem 1.3 for $n = 2$ should also hold.

Conjecture 1: Let $n = 2$ and f satisfy (FA), then there exists a unique convex viscosity solution u to (1.4) *such that* (1.3) *holds for $d = \frac{1}{2\pi}\int_{\mathbb{R}^2}(f-1)$ and $\sigma \in (0, \min\{\beta - 2, 2\})$.*

Conjecture 2: The d^* in Theorem 1.1 *is $\frac{1}{2\pi}\int_{\mathbb{R}^2\setminus D}(f-1) - \frac{1}{2\pi}\text{area}(D)$.*

These two conjectures are closely related in a way that if conjecture one is proved, then conjecture two follows by the same argument in the proof of Theorem 1.1.

The organization of this paper is as follows: First we establish a useful proposition in Sect. 2, which will be used in the proof of all theorems. Then in Sect. 3 we prove Theorem 1.1 using Perron's method. Theorems 1.3 and 1.2 are proved in Sects. 4 and 5, respectively. In the appendix we cite the interior estimates of Caffarelli, Jian and Wang. The proof of all the theorems in this article relies on previous works of Caffarelli [5,7], Jian and Wang [25] and Caffarelli and Li [9]. For example, Caffarelli and Li [9] made it clear that for exterior Dirichlet problems, convex viscosity solutions are strictly convex. On the other hand for Monge–Ampère equations on convex domains, Pogorelov has a well known example of a not-strictly-convex solution. Besides this, we also use the Alexandrov estimates, the interior estimate of Caffarelli [7] and Jian and Wang [25] in an essential way.

2 A useful proposition

Throughout the article we use $B_r(x)$ to denote the ball centered at x with radius r and B_r to denote the ball of radius r centered at 0.

The following proposition will be used in the proof of all theorems.

Proposition 2.1 *Let $R_0 > 0$ be a positive number, $v \in C^0(\mathbb{R}^n\setminus\bar{B}_{R_0})$ be a convex viscosity solution of*

$$det(D^2v) = f_v \quad \mathbb{R}^n\setminus\bar{B}_{R_0}$$

where $f_v \in C^m(\mathbb{R}^n)\setminus\bar{B}_{R_0}$ satisfies

$$\frac{1}{c_0} \leq f_v(x) \leq c_0, \quad x \in \mathbb{R}^n\setminus B_{R_0}$$

and

$$|D^k(f_v(x) - 1)| \leq c_0|x|^{-\beta-k}, \quad |x| > R_0, \quad k = 0, \ldots, m, \ (\beta > 2, m \geq 3). \tag{2.1}$$

Suppose there exists $\epsilon > 0$ such that

$$\left|v(x) - \frac{1}{2}|x|^2\right| \leq c_1|x|^{2-\epsilon}, \quad |x| \geq R_0 \tag{2.2}$$

then for $n \geq 3$, there exist $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ and $C(n, R_0, \epsilon, \beta, c_0, c_1)$ such that,

$$\left|D^k\left(v(x) - \frac{1}{2}|x|^2 - b\cdot x - c\right)\right| \leq C/|x|^{\min\{\beta,n\}-2+k}, \quad |x| > R_1, \ k = 0, \ldots, m+1; \tag{2.3}$$

where $R_1(n, R_0, \epsilon, \beta, c_0, c_1) > R_0$ *depends only on* $n, R_0, \epsilon, \beta, c_0$ *and* c_1. *For* $n = 2$, *there exist* $b \in \mathbb{R}^2$, $d, c \in \mathbb{R}$ *such that for all* $\sigma \in (0, \min\{\beta - 2, 2\})$

$$\left| D^k(v(x) - \frac{1}{2}|x|^2 - b \cdot x - d \log|x| - c) \right| \\ \leq \frac{C(\epsilon, R_0, \beta, c_0, c_1)}{|x|^{\sigma+k}}, \quad |x| > R_1, \ k = 0, \ldots, m+1, \tag{2.4}$$

where $R_1 > R_0$ *depends only on* $\epsilon, R_0, \beta, c_0$ *and* c_1.

Remark 2.1 ϵ may be greater than or equal to 2 in Proposition 2.1.

Proof of Proposition 2.1 Proposition 2.1 is proved in [9] for the case that $f \equiv 1$ outside a compact subset of $\mathbb{R}^n$. For this more general case, Theorem 6.1 in the appendix (A theorem of Caffarelli, Jian and Wang) and Schauder estimates play a central role. First we establish a lemma that holds for all dimensions $n \geq 2$. □

Lemma 2.1 *Under the assumption of Proposition 2.1, let*

$$w(x) = v(x) - \frac{1}{2}|x|^2,$$

then there exist $C(n, R_0, \epsilon, c_0, c_1, \beta) > 0$ *and* $R_1(n, R_0, \epsilon, c_0, c_1, \beta) > R_0$ *such that for any* $\alpha \in (0, 1)$

$$\begin{cases} |D^k w(y)| \leq C|y|^{2-k-\epsilon_\beta}, \quad k = 0, \ldots, m+1, \quad |y| > R_1 \\ \frac{|D^{m+1}w(y_1) - D^{m+1}w(y_2)|}{|y_1 - y_2|^\alpha} \leq C|y_1|^{1-m-\epsilon_\beta-\alpha}, \quad |y_1| > R_1, \ y_2 \in B_{\frac{|y_1|}{2}}(y_1) \end{cases} \tag{2.5}$$

where $\epsilon_\beta = \min\{\epsilon, \beta\}$.

Remark 2.2 The estimate in Lemma 2.1 is much weaker than Proposition 2.1. In order to improve (2.5), ϵ_β will be removed later. However the approach of Lemma 2.1 will be used repeatedly later.

Proof of Lemma 2.1 For $|x| = R > 2R_0$, let

$$v_R(y) = \left(\frac{4}{R}\right)^2 v\left(x + \frac{R}{4}y\right), \quad |y| \leq 2,$$

and

$$w_R(y) = \left(\frac{4}{R}\right)^2 w\left(x + \frac{R}{4}y\right), \quad |y| \leq 2.$$

By (2.2) we have

$$\|v_R\|_{L^\infty(B_2)} \leq C, \quad \|w_R\|_{L^\infty(B_2)} \leq CR^{-\epsilon} \tag{2.6}$$

and

$$v_R(y) - \left(\frac{1}{2}|y|^2 + \frac{4}{R}x \cdot y + \frac{8}{R^2}|x|^2\right) = O(R^{-\epsilon}), \quad B_2.$$

Let $\bar{v}_R(y) = v_R(y) - \frac{4}{R}x \cdot y - \frac{8}{R^2}|x|^2$, clearly $D^2\bar{v}_R = D^2 v_R$. If $R > R_1$ with R_1 sufficiently large, the set

$$\Omega_{1,v} = \{y \in B_2; \quad \bar{v}_R(y) \leq 1\}$$

is between $B_{1.2}$ and B_n. The equation for $\bar{v}_R$ is

$$det(D^2\bar{v}_R(y)) = f_{1,R}(y) := f_v\left(x + \frac{R}{4}y\right), \quad \text{on } B_2. \tag{2.7}$$

Immediately from (2.1) we have, for any $\alpha \in (0, 1)$

$$\|f_{1,R} - 1\|_{L^\infty(B_2)} + \sum_{k=1}^{m-1} \|D^k f_{1,R}\|_{C^\alpha(B_2)} \le CR^{-\beta}. \tag{2.8}$$

Applying Theorem 6.1 on $\Omega_{1,v}$

$$\|D^2 v_R\|_{C^\alpha(B_{1.1})} = \|D^2\bar{v}_R\|_{C^\alpha(B_{1.1})} \le C.$$

Using (2.7) and (2.8) we have

$$\frac{I}{C} \le D^2 v_R \le CI \quad \text{on } B_{1.1} \tag{2.9}$$

for some C independent of R. Then we write (2.7) as

$$a_{ij}^R \partial_{ij} v_R = f_{1,R}, \quad B_2$$

where $a_{ij}^R = cof_{ij}(D^2 v_R)$. Clearly by (2.9) a_{ij}^R is uniformly elliptic and

$$\|a_{ij}^R\|_{C^\alpha(\bar{B}_{1.1})} \le C.$$

Schauder estimate gives

$$\|v_R\|_{C^{2,\alpha}(\bar{B}_1)} \le C(\|v_R\|_{L^\infty(\bar{B}_{1.1})} + \|f_{1,R}\|_{C^\alpha(\bar{B}_2)}) \le C. \tag{2.10}$$

For any $e \in \mathbb{S}^{n-1}$, applying ∂_e to both sides of (2.7), we have

$$a_{ij}^R \partial_{ij}(\partial_e v_R) = \partial_e f_{1,R}. \tag{2.11}$$

Since a_{ij}^R, $\partial_e v_R$ and $\partial_e f_{1,R}$ are bounded in C^α norm, we have

$$\|v_R\|_{C^{3,\alpha}(\bar{B}_1)} \le C, \tag{2.12}$$

which implies

$$\|a_{ij}^R\|_{C^{1,\alpha}(\bar{B}_1)} \le C. \tag{2.13}$$

The difference between (2.7) (with $\bar{v}_R$ replaced by v_R) and $det(I) = 1$ gives

$$\tilde{a}_{ij}\partial_{ij} w_R = f_{1,R}(y) - 1 = O(R^{-\beta}) \tag{2.14}$$

where $\tilde{a}_{ij}(y) = \int_0^1 cof_{ij}(I + tD^2 w_R(y))dt$. By (2.9) and (2.12)

$$\frac{I}{C} \le \tilde{a}_{ij} \le CI, \text{ on } B_{1.1}, \quad \|\tilde{a}_{ij}\|_{C^{1,\alpha}(\bar{B}_1)} \le C.$$

Thus Schauder's estimate gives

$$\|w_R\|_{C^{2,\alpha}(B_1)} \le C(\|w_R\|_{L^\infty(\bar{B}_{1.1})} + \|f_{1,R} - 1\|_{C^\alpha(\bar{B}_1)}) \le CR^{-\epsilon_\beta}. \tag{2.15}$$

Going back to (2.11) and rewriting it as

$$a_{ij}^R \partial_{ij}(\partial_e w_R) = \partial_e f_{1,R}.$$

We obtain, by Schauder's estimate,

$$\|w_R\|_{C^{3,\alpha}(\bar{B}_{1/2})} \le C(\|w_R\|_{L^\infty(\bar{B}_{3/4})} + |Df_{1,R}\|_{C^\alpha(\bar{B}_{3/4})}) \le CR^{-\epsilon_\beta}, \tag{2.16}$$

which immediately implies

$$\|D^3 v_R\|_{C^\alpha(\bar{B}_{1/2})} \le CR^{-\epsilon_\beta}$$

because $\partial_{ije} w_R = \partial_{ije} v_R$. By differentiating on (2.11) with respect to any $e_1 \in \mathbb{S}^{n-1}$ we have

$$a_{ij}^R \partial_{ij}(\partial_{ee_1} w_R) = \partial_{ee_1} f_{1,R} - \partial_{e_1} a_{ij}^R \partial_{ij} \partial_e w_R.$$

(2.16) gives

$$\|\partial_{e_1} a_{ij}^R \partial_{ij} \partial_e w_R\|_{C^\alpha(\bar{B}_{1/2})} \le CR^{-2\epsilon_\beta}.$$

Using (2.8),$\|\partial_{ee_1} f_{1,R}\|_{C^\alpha(\bar{B}_1)} \le CR^{-\beta}$ and Schauder's estimate we have

$$\|w_R\|_{C^{4,\alpha}(\bar{B}_{1/4})} \le CR^{-\epsilon_\beta}. \tag{2.17}$$

Estimates on higher order derivatives can be obtained by further differentiation of the equation and Schauder estimate. (2.5) can be obtained accordingly. Lemma 2.1 is established. □

Next we prove a lemma that improves the estimates in Lemma 2.1.

Lemma 2.2 *Under the same assumptions of Lemma 2.1 and let R_1 be the large constant determined in the proof of Lemma 2.2. If in addition $2\epsilon < 1$, then for $n \ge 3$*

$$\begin{cases} |D^k w(x)| \le C|x|^{2-2\epsilon-k}, \quad |x| > 2R_1, \quad k = 0, \ldots, m+1 \\ \frac{|D^{m+1}w(y_1) - D^{m+1}w(y_2)|}{|y_1 - y_2|^\alpha} \le C|y_1|^{1-m-2\epsilon-\alpha}, \quad |y_1| > 2R_1, \ y_2 \in B_{|y_1|/2}(y_1) \end{cases}$$

where $\alpha \in (0, 1)$. For $n = 2$ and any $\bar{\epsilon} < 2\epsilon < 1$

$$\begin{cases} |D^k w(x)| \le C|x|^{2-\bar{\epsilon}-k}, \quad |x| > 2R_1, \quad k = 0, \ldots, m+1, \\ \frac{|D^{m+1}w(y_1) - D^{m+1}w(y_2)|}{|y_1 - y_2|^\alpha} \le C|y_1|^{1-m-\bar{\epsilon}-\alpha}, \quad |y_1| > 2R_1, \ y_2 \in B_{|y_1|/2}(y_1). \end{cases}$$

Proof of Lemma 2.2 Applying ∂_k to $det(D^2 v) = f_v$ we have

$$a_{ij} \partial_{ij}(\partial_k v) = \partial_k f_v \tag{2.18}$$

where $a_{ij} = cof_{ij}(D^2 v)$. Lemma 2.1 implies

$$|a_{ij}(x) - \delta_{ij}| \le \frac{C}{|x|^\epsilon}, \quad |Da_{ij}(x)| \le \frac{C}{|x|^{1+\epsilon}}, \quad |x| > R_1$$

and for any $\alpha \in (0, 1)$

$$\frac{|Da_{ij}(x_1) - Da_{ij}(x_2)|}{|x_1 - x_2|^\alpha} \le C|x_1|^{-1-\epsilon-\alpha}, \quad |x_1| > 2R_1, \quad x_2 \in B_{|x_1|/2}(x_1).$$

Then applying ∂_l to (2.18) and letting $h_1 = \partial_{kl} v$ we further obtain

$$a_{ij} \partial_{ij} h_1 = \partial_{kl} f_v - \partial_l a_{ij} \partial_{ijk} v.$$

Next we write the equation above as

$$\Delta h_1 = f_2 := \partial_{kl} f_v - \partial_l a_{ij} \partial_{ijk} v - (a_{ij} - \delta_{ij}) \partial_{ij} h_1. \tag{2.19}$$

For any $\alpha \in (0,1)$ the following estimate follows from (2.1) and Lemma 2.1:

$$\begin{cases} |f_2(x)| \le C|x|^{-2-2\epsilon} & |x| \ge 2R_1, \\ \frac{|f_2(x_1)-f_2(x_2)|}{|x_1-x_2|^\alpha} \le \frac{C}{|x_1|^{2+2\epsilon+\alpha}}, & x_2 \in B_{|x_1|/2}(x_1),\ |x_1| \ge 2R_1. \end{cases} \tag{2.20}$$

Note that by Lemma 2.1 $h_1(x) \to \delta_{kl}$ as $x \to \infty$. If $n \ge 3$, we set

$$h_2(x) = -\int\limits_{\mathbb{R}^n \setminus B_{R_1}} \frac{1}{n(n-2)\omega_n} |x-y|^{2-n} f_2(y)dy$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. If $n=2$, we set

$$h_2(x) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2 \setminus B_{R_1}} (\log|x-y| - \log|x|) f_2(y)dy.$$

In either case $\Delta h_2 = f_2$. By elementary estimate it is easy to get

$$|D^j h_2(x)| \le \begin{cases} C|x|^{-2\epsilon-j}, & |x| > 2R_1,\ j=0,1,\ n \ge 3, \\ C|x|^{-\bar{\epsilon}-j}, & |x| > 2R_1,\ j=0,1,\ n=2 \end{cases} \tag{2.21}$$

where $\bar{\epsilon}$ is any positive number less than 2ϵ. Indeed, for each x, let

$$\begin{aligned} E_1 &:= \{y \in \mathbb{R}^n \setminus B_{2R_1}, \quad |y| \le |x|/2, \ \}, \\ E_2 &:= \{y \in \mathbb{R}^n \setminus B_{2R_1}, \quad |y-x| \le |x|/2, \ \}, \\ E_3 &= (\mathbb{R}^n \setminus B_{2R_1}) \setminus (E_1 \cup E_2). \end{aligned}$$

Then it is easy to get (2.21). For the estimate of $D^2 h_2$ we claim that given $\alpha \in (0,1)$, if $n \ge 3$

$$\begin{cases} |D^j h_2(x)| \le C|x|^{-2\epsilon-j}, & j=0,1,2, \quad |x| > 2R_1, \\ \frac{|D^2 h_2(x_1) - D^2 h_2(x_2)|}{|x_1-x_2|^\alpha} \le \frac{C}{|x_1|^{2+2\epsilon+\alpha}}, & x_2 \in B_{\frac{|x_1|}{2}}(x_1),\ |x_1| > 2R_1. \end{cases} \tag{2.22}$$

Replacing 2ϵ by $\bar{\epsilon}$ we get the corresponding estimates of $D^2 h_2$ for $n=2$. The way to obtain (2.22) is standard. Indeed, for each $x_0 \in \mathbb{R}^n \setminus B_{2R_1}$, let $R = |x_0|$, we set

$$h_{2,R}(y) = h_2\left(x_0 + \frac{R}{4}y\right), \quad f_{2,R}(y) = \frac{R^2}{16} f_2\left(x_0 + \frac{R}{4}y\right), \quad |y| \le 2.$$

By (2.20) $\|f_{2,R}\|_{C^\alpha(B_1)} \le CR^{-2\epsilon}$. Therefore Schauder estimate gives

$$\|h_{2,R}\|_{C^{2,\alpha}(B_1)} \le C(\|h_{2,R}\|_{L^\infty(B_2)} + \|f_{2,R}\|_{C^\alpha(B_2)}) \le CR^{-2\epsilon},$$

which is equivalent to (2.22). The way to get the corresponding estimate for $n=2$ is the same. Now we have

$$\Delta(h_1 - h_2) = 0, \quad \mathbb{R}^n \setminus B_{2R_1}.$$

Since we know $h_1 - \delta_{kl} - h_2 \to 0$ at infinity. For $n \ge 3$, by comparing with a multiple of $|x|^{2-n}$ we have

$$|h_1(x) - \delta_{kl} - h_2(x)| \le C|x|^{2-n}, \quad |x| > 2R_1.$$

By the estimate on h_2 we have

$$|h_1(x) - \delta_{kl}| \le C|x|^{-2\epsilon}, \quad |x| > 2R_1.$$

Correspondingly

$$|D^j w(x)| \leq C|x|^{2-j-2\epsilon}, \quad |x| > 2R_1, \quad j = 0, 1, 2, \quad n \geq 3.$$

For $n = 2$ we have

$$|h_1(x) - \delta_{kl} - h_2(x)| \leq C|x|^{-1}, \quad |x| > 2R_1. \tag{2.23}$$

Indeed, let $h_3(y) = h_1(\frac{y}{|y|^2}) - \delta_{kl} - h_2(\frac{y}{|y|^2})$, then $\Delta h_3 = 0$ in $B_{1/2R_1} \setminus \{0\}$ and $\lim_{y \to 0} h_3(y) = 0$. Therefore $|h_3(y)| \leq C|y|$ near 0. (2.23) follows. By fundamental theorem of calculus,

$$|D^j w(x)| \leq C|x|^{2-j-\bar{\epsilon}}, \quad |x| > 2R_1, \quad j = 0, 1, 2, \quad n = 2.$$

Finally we apply Lemma 2.1 to obtain the estimates on higher derivatives. Lemma 2.2 is established. □

Remark 2.3 $m \geq 3$ is used in the estimates of h_2 in the proof of Lemma 2.2.

Now we complete the proof of Proposition 2.1 first for **Case one:** $n \geq 3$.

Let k_0 be a positive integer such that $2^{k_0}\epsilon < 1$ and $2^{k_0+1}\epsilon > 1$ (we choose ϵ smaller if necessary to make both inequalities hold). Let $\epsilon_1 = 2^{k_0}\epsilon$, clearly we have $1 < 2\epsilon_1 < 2$. Applying Lemma 2.2 k_0 times we have

$$\begin{cases} |D^k w(x)| \leq C|x|^{2-\epsilon_1-k}, \quad k = 0, \ldots, m+1, \quad |x| > 2R_1 \\ \frac{|D^{m+1}w(x_1) - D^{m+1}w(x_2)|}{|x_1 - x_2|^\alpha} \leq C|x_1|^{1-m-\epsilon_1-\alpha}, \quad |x_1| > 2R_1, \; x_2 \in B_{|x_1|/2}(x_1). \end{cases} \tag{2.24}$$

Let h_1 and f_2 be the same as in Lemma 2.2. Then we have

$$\begin{cases} |f_2(x)| \leq C|x|^{1-m-2\epsilon_1} + C|x|^{-2-\beta} \quad |x| \geq 2R_1, \\ \frac{|f_2(x_1) - f_2(x_2)|}{|x_1 - x_2|^\alpha} \leq \frac{C}{|x_1|^{m-1+2\epsilon_1+\alpha}} + \frac{C}{|x|^{\beta+2+\alpha}}, \; x_2 \in B_{|x_1|/2}(x_1), \; |x_1| \geq 2R_1. \end{cases}$$

Constructing h_2 as in Lemma 2.2 (the one for $n \geq 3$) we have

$$\begin{cases} |D^j h_2(x)| \leq C|x|^{-2\epsilon_1-j}, \quad j = 0, 1, 2, \quad |x| > 2R_1, \\ \frac{|D^2 h_2(x_1) - D^2 h_2(x_2)|}{|x_1 - x_2|^\alpha} \leq \frac{C}{|x_1|^{2+2\epsilon_1+\alpha}}, \; x_2 \in B_{\frac{|x_1|}{2}}(x_1), \; |x_1| > 2R_1. \end{cases} \tag{2.25}$$

As in the proof of Lemma 2.2 by (2.25) we have

$$|h_1(x) - h_2(x)| \leq C|x|^{2-n}, \quad |x| > 2R_1.$$

Since $2\epsilon_1 > 1$

$$|h_1(x)| \leq |h_2(x)| + C|x|^{2-n} \leq C|x|^{-1}.$$

By Theorem 4 of [20], $\partial_m w(x) \to c_m$ for some $c_m \in \mathbb{R}$ as $|x| \to \infty$. Let $b \in \mathbb{R}^n$ be the limit of ∇w and $w_1(x) = w(x) - b \cdot x$. The equation for w_1 can be written as (for $e \in \mathbb{S}^{n-1}$)

$$a_{ij}\partial_{ij}(\partial_e w_1) = \partial_e f_v. \tag{2.26}$$

By (2.24) the equation above can be written as

$$\Delta(\partial_e w_1) = f_3 := \partial_e f_v - (a_{ij} - \delta_{ij})\partial_{ije} w_1, \quad |x| > 2R_1. \tag{2.27}$$

and we have

$$\begin{cases} |f_3(x)| \leq C(|x|^{-\beta-1} + |x|^{-1-2\epsilon_1}) \leq C|x|^{-1-2\epsilon_1}, \; |x| > 2R_1 \\ \frac{|f_3(x_1) - f_3(x_2)|}{|x_1 - x_2|^\alpha} \leq C|x_1|^{-1-2\epsilon_1-\alpha}, \; |x_1| > 2R_1, \; x_2 \in B_{|x_1|/2}(x_1). \end{cases}$$

Let h_4 solve $\Delta h_4 = f_3$ and the construction of h_4 is similar to that of h_2. Then we have

$$\begin{cases} |D^j h_4(x)| \le C|x|^{1-2\epsilon_1 - j}, \ \ |x| > 2R_1, \ \ j = 0, 1, 2, \\ \frac{|D^2 h_4(x_1) - D^2 h_4(x_2)|}{|x_1 - x_2|^\alpha} \le C|x_1|^{-1-2\epsilon_1-\alpha}, \ \ |x_1| > 2R_1, \ \ x_2 \in B_{|x_1|/2}(x_1). \end{cases}$$

Since $\partial_e w_1 - h_4 \to 0$ at infinity, we have

$$|\partial_e w_1(x) - h_4(x)| \le C|x|^{2-n}, \quad |x| > R_1. \tag{2.28}$$

Therefore we have obtained $|\nabla w_1(x)| \le C|x|^{1-2\epsilon_1}$ on $|x| > R_1$. Using fundamental theorem of calculus

$$|w_1(x)| \le C|x|^{2-2\epsilon_1}, \quad j = 0, 1, \quad |x| > R_1.$$

Lemma 2.1 applied to w_1 gives

$$|D^j w_1(x)| \le C|x|^{2-j-2\epsilon_1}, \quad j = 0.., m+1.$$

Going back to (2.27) we write f_3 as

$$\begin{cases} |f_3(x)| \le C|x|^{-\beta-1} + C|x|^{-1-4\epsilon_1}, \quad |x| > 2R_1, \\ \frac{|f_3(x_1) - f_3(x_2)|}{|x_1 - x_2|^\alpha} \le C(|x_1|^{-\beta-1-\alpha} + |x_1|^{-1-4\epsilon_1-\alpha}), \ |x_1| > 2R_1, \ x_2 \in B_{|x_1|/2}(x_1). \end{cases}$$

The new estimate of h_4 is

$$|h_4(x)| \le C(|x|^{1-\beta} + |x|^{1-4\epsilon_1}), \quad |x| > 2R_1.$$

As before (2.28) holds. Consequently

$$|\nabla w_1(x)| \le C(|x|^{2-n} + |x|^{1-4\epsilon_1}) \le C|x|^{-1}, \quad |x| > 2R_1.$$

By Theorem 4 of [20], $w_1 \to c$ at infinity. Let

$$w_2(x) = w(x) - b \cdot x - c.$$

Then we have $|w_2(x)| \le C$ for $|x| > 2R_1$. Lemma 2.1 applied to w_2 gives

$$|D^k w_2(x)| \le C|x|^{-k}, \quad k = 0, \ldots, m+1, \quad |x| > 2R_1. \tag{2.29}$$

The equation for w_2 can be written as

$$det(I + D^2 w_2(x)) = f_v.$$

Taking the difference between this equation and $det(I) = 1$ we have

$$\tilde{a}_{ij} \partial_{ij} w_2 = f_v - 1, \quad |x| > 2R_1$$

where $\tilde{a}_{ij}$ satisfies

$$|D^j(\tilde{a}_{ij}(x) - \delta_{ij})| \le C|x|^{-2-j}, \quad |x| > 2R_1, \quad j = 0, 1.$$

Using (2.29) this equation can be written as

$$\Delta w_2 = f_4 := f_v - 1 - (\tilde{a}_{ij} - \delta_{ij}) \partial_{ij} w_2, \quad |x| > 2R_1.$$

(2.29) further gives

$$\begin{cases} |f_4(x)| \le C(|x|^{-\beta} + |x|^{-4}), \quad |x| > 2R_1, \\ \frac{|f_4(x_1) - f_4(x_2)|}{|x_1 - x_2|^\alpha} \le C(|x_1|^{-\beta-\alpha} + |x_1|^{-4-\alpha}), \quad |x_1| > 2R_1, \ x_2 \in B_{|x_1|/2}(x_1). \end{cases}$$

Let h_5 be defined similar to h_2. Then h_5 solves $\Delta h_5 = f_4$ in $\mathbb{R}^n \setminus B_{2R_1}$ and satisfies

$$|h_5(x)| \leq C(|x|^{2-\beta} + |x|^{-2}).$$

As before we have

$$|w_2(x) - h_5(x)| \leq C|x|^{2-n}, \quad |x| > 2R_1,$$

which gives

$$|w_2(x)| \leq C(|x|^{2-n} + |x|^{2-\beta} + |x|^{-2}), \quad |x| > 2R_1. \tag{2.30}$$

If $|x|^{-2} > |x|^{2-n} + |x|^{2-\beta}$ we can apply the same argument as above finite times to remove the $|x|^{-2}$ from (2.30). Eventually by Lemma 2.1 we have (2.3). Proposition 2.1 is established for $n \geq 3$.

Finally we prove **Case two:** $n = 2$.

As in the case for $n \geq 3$ we let k_0 be a positive integer such that $2^{k_0}\epsilon < 1$ and $2^{k_0+1}\epsilon > 1$ (we choose ϵ smaller if necessary to make both inequalities hold). Let $\epsilon_1 < 2^{k_0}\epsilon$ and we let $1 < 2\epsilon_1 < 2$. Applying Lemma 2.2 k_0 times then (2.24) holds. Then we consider the equation for w. By taking the difference between the equation for v and $det(I) = 1$ we have

$$\tilde{a}_{ij}\partial_{ij} w = f_v - 1.$$

We further write the equation above as

$$\Delta w = f_5 := f_v - 1 - (\tilde{a}_{ij} - \delta_{ij})\partial_{ij} w.$$

By (2.24)

$$|f_5(x)| \leq C|x|^{-2\epsilon_1}, \quad |x| > R_1.$$

Let

$$h_6(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2 \setminus B_{R_1}} (\log|x - y| - \log|x|) f_5(y) dy.$$

Then elementary estimate gives

$$|h_6(x)| \leq C|x|^{\epsilon_2}, \quad |x| > R_1$$

for some $\epsilon_2 \in (0, 1)$. Since $w - h_6$ is harmonic on $\mathbb{R}^2 \setminus B_{R_1}$ and $w - h_6 = O(|x|^{2-\epsilon_1})$, there exist $b \in \mathbb{R}^2$ and $d_1, d_2 \in \mathbb{R}$ such that

$$w(x) - h_6(x) = b \cdot x + d_1 \log|x| + d_2 + O(1/|x|) \quad |x| > 2R_1. \tag{2.31}$$

Equation (2.31) is standard. For the convenience of the readers we include the proof. Let $z_l(r)$ be the projection of $w - h_6$ on $\sin l\theta$ for $l = 1, 2, \ldots$. Then z_l satisfies

$$z_l''(r) + \frac{1}{r} z_l'(r) - \frac{l^2}{r^2} z_l(r) = 0, \quad r > 2R_1.$$

Clearly $z_l(r) = c_{1l} r^l + c_{2l} r^{-l}$. Since $z_l(r) \leq C r^{2-\epsilon_1}$ we have $c_{1l} = 0$ for all $l \geq 2$. Thus $z_l(r) = c_{2l} r^{-l}$. Let C be a constant such that $\max_{B_{2R_1}} |w - h_6| \leq C$. Then $|z_l(2R_1)| \leq C$, which gives $|c_{2l}| \leq C(2R_1)^l$. The estimate for the projection of $w - h_6$ over $\cos l\theta$ for $l \geq 2$ is the same. The term $d_1 \log|x| + d_2$ comes from the projection onto 1. The projection onto $\cos\theta$ and $\sin\theta$ gives $b \cdot x$. (2.31) is established.

Let

$$w_1(x) = w(x) - b \cdot x.$$

Then it holds $|w_1(x)| \le C|x|^{\epsilon_2}$. Lemma 2.1 gives

$$|D^k w_1(x)| \le C|x|^{\epsilon_2 - k}, \quad k = 0, \dots, m+1, \quad |x| > 2R_1.$$

The equation for w_1 can be written as

$$\Delta w_1 = O(|x|^{-\beta}) + O(|x|^{2\epsilon_2 - 4}).$$

Let

$$h_7(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2 \setminus B_{2R_1}} (\log|x-y| - \log|x|)\Delta w_1(y)dy.$$

Then

$$|h_7(x)| \le C(|x|^{2-\beta+\epsilon} + |x|^{2\epsilon_2 - 2 + \epsilon})$$

for $\epsilon > 0$ arbitrarily small. Since $w_1 - h_7$ is harmonic on $\mathbb{R}^2 \setminus B_{2R_1}$ and $w_1(x) - h_7(x) = O(|x|^{\epsilon_2})$, we have, for some $d, c \in \mathbb{R}$

$$w_1(x) - h_7(x) = d\log|x| + c + O(1/|x|).$$

Using the estimates on h_7 we have

$$w_1(x) = d\log|x| + c + O(|x|^{2\epsilon_2 - 2 + \epsilon}) + O(|x|^{2-\beta+\epsilon}). \tag{2.32}$$

To obtain (2.4) we finally let

$$v_1(x) = v(x) - b \cdot x - c$$

and

$$H(x) = \frac{1}{2}|x|^2 + d\log|x|.$$

Clearly $det(D^2 v_1(x)) = f_v(x)$ and $det(D^2 H(x)) = 1 - \frac{d^2}{|x|^4}$. Let $w_2(x) = v_1(x) - H(x)$. By (2.32) we already have

$$|w_2(x)| \le C|x|^{-\epsilon_3}, \quad |x| > 2R_1$$

for some $\epsilon_3 > 0$. Using Theorem 6.1 as well as Schauder estimate as in the proof of Lemma 2.1 we obtain

$$|D^k w_2(x)| \le C|x|^{-\epsilon_3 - k} \quad |x| > 2R_1, \quad k = 0, 1, 2. \tag{2.33}$$

Thus the equation of w_2 can be written as

$$\Delta w_2(x) = O(|x|^{-4-2\epsilon_3}) + O(|x|^{-\beta}).$$

Let

$$h_8(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2 \setminus B_{R_1}} (\log|x-y| - \log|x|)\Delta w_2(y)dy.$$

Then

$$|D^j h_8(x)| = O(|x|^{-2-j} + |x|^{\epsilon_5+2-\beta-j}), \quad j=0,1, \quad |x|>R_1 \tag{2.34}$$

for all $\epsilon_5 > 0$. Then we have $w_2(x) - h_8(x) = O(|x|^{-2})$ because of (2.33), (2.34) and the argument in the proof of (2.31). Consequently

$$w_2(x) = O(|x|^{-2} + |x|^{\epsilon_6+2-\beta}), \quad |x|>R_1$$

for all $\epsilon_6 > 0$. The estimates on higher derivatives of w_2 can be obtained by Lemma 2.1. Proposition 2.1 is established for $n=2$ as well. □

3 Proof of Theorem 1.1

Without loss of generality we assume that $B_2 \subset D \subset B_{\bar{r}}$. First we prove a lemma that will be used in the proof for $n \geq 3$ and $n=2$.

Lemma 3.1 *There exists $c_1(n,\phi,D)$ such that for every $\xi \in \partial D$, there exists w_ξ such that*

$$\begin{cases} det(D^2 w_\xi(x)) \geq f(x) \quad \mathbb{R}^n \backslash D, \\ w_\xi(\xi) = \phi(\xi), \quad w_\xi(x) < \phi(x), \quad \forall x \in \partial D, \ x \neq \xi, \\ w_\xi(x) \leq \frac{1}{2}|x|^2 + c_1, \quad x \in (\mathbb{R}^n \backslash D) \cap B(0, 10 diam(D)). \end{cases}$$

Proof of Lemma 3.1 Let f_1 be a smooth radial function on $\mathbb{R}^n$ such that $f_1 > f$ on $\mathbb{R}^n \backslash D$ and f_1 satisfies (FA). Let

$$z(x) = \int_0^{|x|} \left(\int_0^s n t^{n-1} f_1(t) dt \right)^{\frac{1}{n}} ds.$$

Then $det(D^2 z(x)) = f_1(x)$ on $\mathbb{R}^n$ and

$$|z(x) - \frac{1}{2}|x|^2| \leq \begin{cases} C, \quad n \geq 3, \\ C\log(2+|x|), \quad n=2 \end{cases} \quad x \in \mathbb{R}^n.$$

Since D is strictly convex, we can put ξ as the origin using a translation and a rotation and then assume that D stays in $\{x_n > 0\}$. Let $x_n = \rho(x')$ ($x' = (x_1, \ldots, x_{n-1})$) be the part of boundary around ξ. By the strict convexity we assume

$$\rho(x') = \frac{1}{2} \sum_{1 \leq \alpha, \beta \leq n-1} B_{\alpha\beta} x_\alpha x_\beta + o(|x'|^2)$$

where $(B_{\alpha\beta}) \geq \delta I$ for some $\delta > 0$. By subtracting a linear function from z we obtain z_ξ that satisfies

$$\begin{cases} det(D^2 z_\xi) \geq f, \quad \mathbb{R}^n \backslash D, \\ z_\xi(0) = \phi(\xi), \quad \nabla z_\xi(0) = \nabla \phi(\xi) \end{cases}$$

and

$$\left| z_\xi(x) - \frac{1}{2}|x|^2 \right| \leq C|x|, \quad x \in \mathbb{R}^n \backslash D.$$

Next we further adjust z_ξ by defining

$$w_\xi(x) = z_\xi(x) - A_\xi x_n$$

for A_ξ large to be determined. When evaluated on ∂D near 0,

$$w_\xi(x', \rho(x')) - \phi(x', \rho(x')) \le C|x'|^2 - A_\xi \rho(x').$$

Therefore for $|x'| \le \delta_1$ for some δ_1 small we have $w_\xi(x', \rho(x')) < \phi(x', \rho(x'))$. For $|x'| > \delta_1$, the convexity of ∂D yields

$$x_n \ge \delta_1^3, \quad \forall x \in \partial D \backslash \{(x', \rho(x')) : \quad |x'| < \delta_1.\}.$$

Then by choosing A_ξ possibly larger (but still under control) we have $w_\xi(x) < \phi(x)$ for all $x \in \partial D$. Clearly A_ξ has a uniform bound for all $\xi \in \partial\Omega$. Lemma 3.1 is established. □

Let

$$\underline{w}(x) = \max\left\{w_\xi(x) \;\middle|\; \xi \in \partial D\right\}.$$

It is clear by Lemma 3.1 that $\underline{w}$ is a locally Lipschitz function in $B_{2\bar r}\backslash D$, and $\underline{w} = \varphi$ on ∂D. Since w_ξ is a smooth convex solution of (1.1), $\underline{w}$ is a viscosity subsolution of (1.1) in $B_{2\bar r}\backslash \overline{D}$. Let c_1 be the constant determined in Lemma 3.1. Then we have

$$\underline{w}(x) \le \frac{1}{2}|x|^2 + c_1, \quad B_{2\bar r}\backslash \bar D.$$

We finish the proof of Theorem 1.1 in two cases.

Case one: $n \ge 3$. Clearly we only need to prove the existence of solutions for $A = I$ and $b = 0$, as the general case can be reduced to this case by a linear transformation. Let $\bar f$ and $\underline{f}$ be smooth, radial functions such that $\underline{f} < f < \bar f$ in $\mathbb{R}^n \backslash D$ and suppose $\underline{f}$ and $\bar f$ satisfy (FA). For $d > 0$ and $\beta_1, \beta_2 \in \mathbb{R}$, set

$$\underline{u}_d(x) = \beta_1 + \int_{\bar r}^{r} \left(\int_1^s nt^{n-1}\bar f(t)dt + d\right)^{\frac{1}{n}} ds, \quad r = |x| > 2,$$

and

$$\overline{u}_d(x) = \beta_2 + \int_{2}^{r} \left(\int_1^s nt^{n-1}\underline{f}(t)dt + d\right)^{\frac{1}{n}} ds, \quad r = |x| > 2.$$

Clearly

$$\det(D^2 \underline{u}_d) = \bar f \ge f, \quad \mathbb{R}^n \backslash \bar D,$$

and

$$\det(D^2 \overline{u}_d) = \underline{f} \le f, \quad \mathbb{R}^n \backslash \bar D.$$

On the other hand,

$$\underline{u}_d(x) \le \beta_1, \quad \text{in } B_{\bar r}\backslash \overline{D}, \ \forall\, d > 0. \tag{3.1}$$

and

$$\overline{u}_d(x) \ge \beta_2, \quad \text{in } B_{\bar r}\backslash \overline{D}, \ \forall\, d > 0. \tag{3.2}$$

Let

$$\beta_1 := \min\{\underline{w}(x) \;|\; x \in \overline{B_{\bar r}}\backslash D\} - 1 < \min_{\partial D} \varphi,$$
$$\beta_2 := \max_{\partial D} \varphi + 1.$$

This shows that $\underline{u}_d$ and $\overline{u}_d$ are continuous convex subsolution and supersolution of (1.5), respectively. By the definition of $\underline{u}_d$ by choosing d large enough, say $d \geq d_0$, we can make

$$\underline{u}_d > \underline{w}(x) + 1, \quad |x| = \bar{r} + 1.$$

By (3.1) and the above, the function

$$u_{1,d}(x) = \begin{cases} \underline{u}_d, & |x| \geq \bar{r} + 1, \\ \underline{w}(x), & x \in B_{\bar{r}} \backslash D, \\ \max\{\underline{w}(x), \underline{u}_d\}, & x \in B_{\bar{r}+1} \backslash B_{\bar{r}} \end{cases}$$

is a viscosity subsolution of (1.5) if $d \geq d_0$.

Next we consider the asymptotic behavior of $\underline{u}_d$ and $\bar{u}_d$ when d is fixed. Using (FA) it is easy to obtain

$$\underline{u}_d(x) = \frac{1}{2}|x|^2 + \mu_1(d) + O(|x|^{2-\min\{\beta,n\}}),$$

and

$$\overline{u}_d(x) = \frac{1}{2}|x|^2 + \mu_2(d) + O(|x|^{2-\min\{\beta,n\}}),$$

where

$$\mu_1(d) = \beta_1 - \frac{\bar{r}^2}{2} + \int_{\bar{r}}^{\infty} \left(\left(\int_1^s nt^{n-1} \bar{f}(t)dt + d \right)^{\frac{1}{n}} - s \right) ds,$$

and

$$\mu_2(d) = \beta_2 - 2 + \int_2^{\infty} \left(\left(\int_1^s nt^{n-1} \underline{f}(t)dt + d \right)^{\frac{1}{n}} - s \right) ds.$$

It is easy to see that $\mu_1(d)$ and $\mu_2(d)$ are strictly increasing functions of d and

$$\lim_{d\to\infty} \mu_1(d) = \infty, \quad \text{and} \quad \lim_{d\to\infty} \mu_2(d) = \infty. \tag{3.3}$$

Let $c_* = \mu_1(d_0)$, recall that for $d > d_0$, $u_{1,d}$ is a viscosity subsolution. For every $c > c_*$, there exists a unique $d(c)$ such that

$$\mu_1(d(c)) = c. \tag{3.4}$$

So $\underline{u}_{d(c)}$ satisfies

$$\underline{u}_{d(c)}(x) = \frac{1}{2}|x|^2 + c + O(|x|^{2-\min\{\beta,n\}}), \quad \text{as} \quad x \to \infty. \tag{3.5}$$

Also there exists $d_2(c)$ such that $\mu_2(d_2(c)) = c$ and

$$\overline{u}_{d_2(c)}(x) = \frac{1}{2}|x|^2 + c + O(|x|^{2-\min\{\beta,n\}}), \quad \text{as} \quad x \to \infty. \tag{3.6}$$

By (3.5) and (3.6)

$$\lim_{|x|\to\infty} (\underline{u}_{d(c)}(x) - \overline{u}_{d_2(c)}(x)) = 0.$$

On the other hand, by the definition of β_1 we have $\bar{u}_{d_2(c)} > u_{1,d(c)}$ on ∂D. Thus, in view of the comparison principle for smooth convex solutions of Monge–Ampère, (see [11]), we have

$$u_{1,d(c)} \leq \bar{u}_{d_2(c)}, \quad \text{on } \mathbb{R}^n \backslash D. \tag{3.7}$$

For any $c > c_*$, let $\mathcal{S}_c$ denote the set of $v \in C^0(\mathbb{R}^n \backslash D)$ which are viscosity subsolutions of (1.5) in $\mathbb{R}^n \backslash \overline{D}$ satisfying

$$v = \varphi, \quad \text{on } \partial D, \tag{3.8}$$

and

$$u_{1,d(c)} \leq v \leq \bar{u}_{d_2(c)}, \quad \text{in } \mathbb{R}^n \backslash D. \tag{3.9}$$

We know that $u_{1,d(c)} \in \mathcal{S}_c$. Let

$$u(x) := \sup \{v(x) \mid v \in \mathcal{S}_c\}, \quad x \in \mathbb{R}^n \backslash D.$$

Then u is convex and of class $C^0(\mathbb{R}^n \backslash D)$. By (3.5), and the definitions of $u_{1,d(c)}$ and $\bar{u}_{d_2(c)}$

$$u(x) \geq u_{1,d(c)}(x) = \frac{1}{2}|x|^2 + c + O(|x|^{2-\min\{\beta,n\}}), \quad \text{as} \quad x \to \infty \tag{3.10}$$

and

$$u(x) \leq \bar{u}_{d_2(c)}(x) = \frac{1}{2}|x|^2 + c + O(|x|^{2-\min\{\beta,n\}}).$$

The estimate (1.2) for $k = 0$ follows.

Next, we prove that u satisfies the boundary condition. It is obvious from the definition of $u_{1,d(c)}$ that

$$\liminf_{x \to \xi} u(x) \geq \lim_{x \to \xi} u_{1,d(c)}(x) = \varphi(\xi), \quad \forall\, \xi \in \partial D.$$

So we only need to prove that

$$\limsup_{x \to \xi} u(x) \leq \varphi(\xi), \quad \forall\, \xi \in \partial D.$$

Let $\omega_c^+ \in C^2(\overline{B_{\bar{r}} \backslash D})$ be defined by

$$\begin{cases} \Delta \omega_c^+ = 0, & \text{in } B_{\bar{r}+1} \backslash \overline{D}, \\ \omega_c^+ = \varphi, & \text{on } \partial D, \\ \omega_c^+ = \max\limits_{\partial B_{\bar{r}+1}} \bar{u}_{d_2(c)}, & \text{on } \partial B_{\bar{r}+1}. \end{cases}$$

It is easy to see that a viscosity subsolution v of (1.5) satisfies $\Delta v \geq 0$ in viscosity sense. Therefore, for every $v \in \mathcal{S}_c$, by $v \leq \omega_c^+$ on $\partial(B_{\bar{r}} \backslash D)$, we have

$$v \leq \omega_c^+ \quad \text{in } B_{\bar{r}} \backslash \overline{D}.$$

It follows that

$$u \leq \omega_c^+ \quad \text{in } B_{\bar{r}} \backslash \overline{D},$$

and then

$$\limsup_{x \to \xi} u(x) \leq \lim_{x \to \xi} \omega_c^+(x) = \varphi(\xi), \quad \forall\, \xi \in \partial D.$$

Finally, we prove u is a solution of (1.1). For $\bar{x} \in \mathbb{R}^n \backslash \overline{D}$, fix some $\epsilon > 0$ such that $B_\epsilon(\bar{x}) \subset \mathbb{R}^n \backslash \overline{D}$. By the definition of u, $u \leq \bar{u}$. We claim that there is a convex viscosity solution to $\widetilde{u} \in C^0(\overline{B_\epsilon(\bar{x})})$ to

$$\begin{cases} \det(D^2 \widetilde{u}) = f, & x \in B_\epsilon(\bar{x}), \\ \widetilde{u} = u, & x \in \partial B_\epsilon(\bar{x}). \end{cases}$$

Indeed, let ϕ_k be a sequence of smooth functions on $\partial B_\epsilon(\bar{x})$ satisfying

$$u \leq \phi_k \leq u + \frac{1}{k}.$$

Let f_k be a sequence of smooth positive functions tending to f and $f_k \leq f$. Let ψ_k be the convex solution to

$$\begin{cases} det(D^2 \psi_k) = f_k & B_\epsilon(\bar{x}), \\ \psi_k = \phi_i \quad \text{on} & \partial B_\epsilon(\bar{x}). \end{cases}$$

Clearly $\psi_k \geq u$. On the other hand, let h_k be the harmonic function on $B_\epsilon(\bar{x})$ with $h_k = \phi_k$ on $\partial B_\epsilon(\bar{x})$. Then we have $u_k \leq h_k$. Therefore $|\psi_k|$ is uniformly bounded over any compact subset of $B_\epsilon(\bar{x}).|\nabla \psi_k|$ is also uniformly bounded over all compact subsets of $B_\epsilon(\bar{x})$ by the convexity. Thus ψ_k converges along a subsequence to $\tilde{u}$ in $B_\epsilon(\bar{x})$. By the closeness between h_k to u on $\partial B_\epsilon(\bar{x})$, $\tilde{u}$ can be extended as a continuous function to $\bar{B}_\epsilon(\bar{x})$. By the maximum principle, $u \leq \widetilde{u} \leq \bar{u}_{d_2(c)}$ on B_ϵ. Define

$$w(y) = \begin{cases} \widetilde{u}(y), & \text{if } y \in B_\epsilon, \\ u(y), & \text{if } y \in \mathbb{R}^2 \backslash (D \cup B_\epsilon(\bar{x})). \end{cases}$$

Clearly, $w \in \mathcal{S}_c$. So, by the definition of u, $u \geq w$ on $B_\epsilon(\bar{x})$. It follows that $u \equiv \widetilde{u}$ on $B_\epsilon(\bar{x})$. Therefore u is a viscosity solution of (1.1). We have proved (1.2) for $k = 0$. The estimates of derivatives follow from Proposition 2.1. Theorem 1.1 is established for $n \geq 3$.

Case two: $n = 2$.

As in case one we let $\bar{f}$ be a radial function such that $\bar{f}(|x|) \geq f(x)$ in $\mathbb{R}^2 \backslash D$, and $\bar{f}$ also satisfies (FA). Let

$$\underline{u}_d(x) = \beta_1 + \int_{\bar{r}}^{r} \left(\int_1^s 2t \bar{f}(t) dt + d \right)^{\frac{1}{2}} ds$$

for $d \geq 0$ and $r > 1$. Here we choose $\beta_1 = \min_{\partial D} \phi - 1$. Clearly

$$\underline{u}_d(x) < \underline{w}(x) \quad B_{\bar{r}} \backslash D, \quad \forall d \geq 0.$$

Then we choose d^* large so that for all $d \geq d^*$, $\underline{u}_d(x) > \underline{w}(x)$ on $\partial B_{\bar{r}+1}$. Let

$$u_{1,d}(x) = \begin{cases} \underline{w}(x), & B_{\bar{r}} \backslash \bar{D} \\ \max\{\underline{w}(x), \underline{u}_d\}, & B_{\bar{r}+1} \backslash \bar{B}_{\bar{r}}, \\ \underline{u}_d, & \mathbb{R}^2 \backslash B_{\bar{r}+1}. \end{cases}$$

Then $u_{1,d}$ is a convex viscosity subsolution of (1.5). Let

$$A_d = d - 1 + \int_1^\infty 2t(\bar{f}(t) - 1) dt.$$

Then elementary computation gives

$$\underline{u}_d(x) = \frac{1}{2}|x|^2 + A_d \log|x| + O(1).$$

Next we let $\underline{f}$ be a radial function such that $\underline{f}(|x|) \le f(x)$ for $x \in \mathbb{R}^2\backslash\bar{D}$. Suppose $\underline{f}$ also satisfies (FA) and is positive and smooth on $\mathbb{R}^2$. Let

$$\bar{u}_d(x) = \beta_2 + \int_2^r \left(\int_1^s 2t\underline{f}(t)dt + d \right)^{\frac{1}{2}} ds.$$

Let

$$L_d = d - 1 + \int_1^\infty 2t(\underline{f}(t) - 1)dt.$$

Then the asymptotic behavior of $\bar{u}_d$ at infinity is

$$\bar{u}_d(x) = \frac{1}{2}|x|^2 + L_d \log|x| + O(1).$$

Thus for all $d > d^*$, we can choose d_1 such that $L_{d_1} = A_d$. Then we choose β_1 such that $\bar{u}_{d_1} > \phi$ on ∂D and $\bar{u}_{d_1} > \underline{u}_d$ at infinity. As in case one, by taking the supremum of subsolutions we obtain a solution u that is equal to ϕ on ∂D and

$$u(x) = \frac{1}{2}|x|^2 + A_d \log|x| + O(1).$$

By Proposition 2.1

$$u(x) = \frac{1}{2}|x|^2 + A_d \log|x| + c + o(1).$$

The following lemma says the constant term is uniquely determined by other parameters.

Lemma 3.2 *Let u_1, u_2 be two locally convex smooth functions on $\mathbb{R}^2\backslash\bar{D}$ where D satisfies the same assumption as in Theorem 1.1. Suppose u_1 and u_2 both satisfy*

$$\begin{cases} det(D^2u) = f \ in \quad \mathbb{R}^2\backslash\bar{D}, \\ u = \phi, \quad on \quad \partial D \end{cases}$$

with f satisfying (FA) and for the same constant d

$$u_i(x) - \frac{1}{2}|x|^2 - d\log|x| = O(1), \quad x \in \mathbb{R}^2\backslash\bar{D}, \quad i = 1, 2. \tag{3.11}$$

Then $u_1 \equiv u_2$.

Proof of Lemma 3.2 By Proposition 2.1 we see that when (3.11) holds, we have

$$D^2u_i(x) = I + O(|x|^{-2+\epsilon}), \quad i = 1, 2$$

for $\epsilon > 0$ small and $|x|$ large. For the proof of this lemma we only need

$$D^2u_i(x) = I + O(|x|^{-\frac{3}{2}}), \quad |x| > 1, \quad i = 1, 2. \tag{3.12}$$

By Proposition 2.1,

$$u_i(x) = \frac{1}{2}|x|^2 + d\log|x| + c_i + O(1/|x|^\sigma), \quad i = 1, 2$$

for $\sigma \in (0, \min\{\beta - 2, 2\})$. Without loss of generality we assume $c_1 > c_2$. If $c_1 = c_2$ we know $u_1 \equiv u_2$ by maximum principle. Since $u_1 = u_2$ on ∂D, we have, $u_1 > u_2$ in $\mathbb{R}^2 \backslash \bar{D}$. Let $w = u_1 - u_2$, then w satisfies

$$a_{ij}\partial_{ij}w = 0, \quad \mathbb{R}^2 \backslash \bar{D}$$

where

$$a_{ij}(x) = \int_0^1 cof_{ij}(tD^2u_1 + (1-t)D^2u_2)dt.$$

By the assumption of Lemma 3.2 and (3.12), a_{ij} is uniformly elliptic and

$$a_{ij}(x) = \delta_{ij} + O(|x|^{-\frac{3}{2}}), \quad x \in \mathbb{R}^2 \backslash \bar{D}. \tag{3.13}$$

Let $a_0 < \frac{1}{2}a_1$ be positive constants to be determined. We set $h_\epsilon = \epsilon\log(|x| - a_0)$ over $a_1 < |x| < \infty$. Direct computation shows, by (3.13) that

$$\begin{aligned} a_{ij}\partial_{ij}h_\epsilon &= \Delta h_\epsilon + (a_{ij} - \delta_{ij})\partial_{ij}h_\epsilon \\ &\le -\frac{\epsilon a_0}{(|x| - a_0)^2|x|} + C\epsilon|x|^{-7/2}, \\ &\le -\frac{4\epsilon a_0}{|x|^3} + C\epsilon|x|^{-7/2}, \quad |x| > a_1 > a_0. \end{aligned}$$

By choosing a_0 sufficiently large and $a_1 > 2a_0$ we have

$$a_{ij}\partial_{ij}h_\epsilon < 0, \quad a_1 < |x| < \infty.$$

Let $R > a_1$ and $M_R = \max_{|x|=R} w$. Let $v = w - M_R$, then clearly for all $\epsilon > 0$, h_ϵ is greater than v on ∂B_R and at infinity. Thus for any compact subset $K \subset\subset \mathbb{R}^2 \backslash \bar{B}_R$, $v < h_\epsilon$. Let $\epsilon \to 0$ we have

$$w(x) \le M_R, \quad \forall |x| \ge R.$$

Taking any $R_1 > R$, we have $\max_{|x|=R_1} w \le M_R$. Strong maximum principle implies that either $\max_{|x|=R_1} w < M_R$ for all $R_1 > R$ or w is a constant. w is not a constant, therefore we have $\max_{|x|=R_1} w < M_R$ for all $R_1 > R$. However, this means over the region $B_{R_1} \backslash \bar{D}$, the maximum of w is attained at an interior point, a contradiction to the elliptic equation that w satisfies. Thus Lemma 3.2 is established. □

Lemma 3.2 uniquely determines the constant in the expansion, then by Proposition 2.1 we obtain (1.3). Thus Theorem 1.1 for the case $n = 2$ is established. □

4 Proof of Theorem 1.3

We only need to consider the existence part as the uniqueness part follows immediately from maximum principles. For the existence part we only need to consider the case that

$A = I, b = 0$ and $c = 0$, because the general case can be reduced to this case by a linear transformation. Consider u_R that solves

$$\begin{cases} det(D^2u_R) = f, & B_R, \\ u_R = \frac{R^2}{2}, & \partial B_R. \end{cases} \tag{4.1}$$

We shall bound u_R above and below by two radial functions. Let h be a smooth radial function, then at the point $(|x|, 0, ..., 0)$

$$D^2h(x) = \text{diag}(h''(r), h'(r)/r, ..., h'(r)/r), \quad r = |x|.$$

Thus $det(D^2h)(x) = h''(r)(h'(r)/r)^{n-1}$.

We first construct a subsolution $h_-(r)$: Let $\bar{f}$ be a radial function such that $\bar{f} > f$ and $\bar{f}$ satisfies (FA).

$$h_-(r) = \int_0^r \left(\int_0^s nt^{n-1}\bar{f}(t)dt \right)^{\frac{1}{n}} ds.$$

Clearly $det(D^2h_-) = \bar{f}$ in $\mathbb{R}^n$ and since $\bar{f}(t) = 1 + O(t^{-\beta})$ it is easy to verify that

$$h_-(r) = \frac{1}{2}|x|^2 + O(1).$$

Next we construct a super solution. Let $\underline{f}$ be a radial function less than $f(x)$ and $\underline{f}$ also satisfy (FA),

$$h_+(r) = \int_0^r \left(\int_0^s nt^{n-1}\underline{f}(t)dt \right)^{\frac{1}{n}} ds.$$

Similarly we have $det(D^2h_+) = \underline{f}$ in $\mathbb{R}^n$ and $h_+(r) = \frac{1}{2}r^2 + O(1)$ for r large. Let β_- be a constant such that $h_-(|x|) + \beta_- \le \frac{1}{2}|x|^2$, β_+ be a constant such that $h_+(|x|) + \beta_+ \ge \frac{1}{2}|x|^2$. Then by maximum principle

$$h_-(r) + \beta_- \le u_R(x) \le h_+(r) + \beta_+, \quad |x| \le R. \tag{4.2}$$

Let $R \to \infty$ and the sequence u_R converges to a global solution u that satisfies $det(D^2u) = f$ in $\mathbb{R}^n$ and $u - \frac{1}{2}|x|^2 = O(1)$. For this convergence, we use the fact that for any $K \subset\subset \mathbb{R}^n$, $|u_R(x) - \frac{1}{2}|x|^2| \le C(K)$ and by Caffarelli's $C^{1,\alpha}$ estimate [6], $\|\nabla u_R\|_{L^\infty(K)} \le C(K)$. Thus u_R converges to a convex viscosity solution u to $det(D^2u) = f$ in $\mathbb{R}^n$ with the property that

$$\left| u(x) - \frac{1}{2}|x|^2 \right| \le C, \quad \mathbb{R}^n.$$

By Proposition 2.1, there exists a $c^* \in \mathbb{R}$ such that

$$\lim_{|x|\to\infty} |x|^{\min\{\beta,n\}-2+k} \left(D^k(u(x) - \frac{1}{2}|x|^2 - c^*) \right) < \infty$$

for $k = 0, \ldots, m + 1$. After a translation the solution with the desired asymptotic behavior can be found. Theorem 1.3 is established. □

5 The proof of Theorem 1.2

Without loss of generality we assume $u(0) = 0 = \min_{\mathbb{R}^n} u$. The goal is to show that there exists a linear transformation T such that $v = u \cdot T$ satisfies (2.2). Then we employ Proposition 2.1 to finish the proof. The proof of v satisfying (2.2) is by the argument of Caffarelli and Li.

Suppose $c_0^{-1} \le \inf_{\mathbb{R}^n} f \le \sup_{\mathbb{R}^n} f < c_0$, only under this assumption it is proved in [9] that for M large and

$$\Omega_M := \{x \in \mathbb{R}^n; \quad u(x) < M\}$$

there exists $a_M \in \mathcal{A}$ such that

$$B_{R/C} \subset a_M(\Omega_M) \subset B_{CR}, \tag{5.1}$$

where $R = \sqrt{M}$ and $C > 1$ is a constant independent of M. Let

$$O := \{y; \quad a_M^{-1}(Ry) \in \Omega_M\}.$$

Then $B_{1/C} \subset O \subset B_C$. Set

$$\xi(y) := \frac{1}{R^2} u(a_M^{-1}(Ry)),$$

then we have

$$\begin{cases} det(D^2\xi) = f(a_M^{-1}(Ry)), & \text{in} \quad O, \\ \xi = 1, & \text{on} \quad \partial O. \end{cases} \tag{5.2}$$

Let $\bar{\xi}$ solve

$$\begin{cases} det(D^2\bar{\xi}) = 1, & \text{in} \quad O, \\ \bar{\xi} = 1, & \text{on} \quad \partial O. \end{cases}$$

By Pogorelov's estimate

$$\frac{1}{C} I \le D^2\bar{\xi} \le CI, \quad |D^3\bar{\xi}(x)| \le C, \ \ x \in O, \ \ dist(x, \partial O) \ge \delta.$$

We claim that there exists $C > 0$ independent of M such that

$$|\xi(x) - \bar{\xi}(x)| \le C/R, \quad x \in O. \tag{5.3}$$

Indeed, by the Alexandrov estimate ([8])

$$-\min_{\bar{O}}(\xi - \bar{\xi}) \le C\left(\int_{S^+} det(D^2(\xi - \bar{\xi}))\right)^{1/n}$$

where

$$S^+ := \{x \in O; \quad D^2(\xi - \bar{\xi}) > 0 \quad \}.$$

On S^+

$$\frac{D^2\xi}{2} = \frac{D^2(\xi - \bar{\xi})}{2} + \frac{D^2\bar{\xi}}{2},$$

so the concavity of $det^{\frac{1}{n}}$ on positive definite symmetric matrices implies

$$det(D^2(\xi - \bar{\xi}))^{\frac{1}{n}} \le f(a_M^{-1}(Ry))^{\frac{1}{n}} - 1.$$

Thus

$$-\min_{\bar{O}}(\xi - \bar{\xi}) \le C\Big(\int_{S^+} |f(a_M^{-1}(Ry))^{\frac{1}{n}} - 1|^n dy\Big)^{\frac{1}{n}}.$$

Let $z = a_M^{-1}(Ry)$, i.e. $a_M z = Ry$ then $dz = R^n dy$

$$\left(\int_{S^+} |f(a_M^{-1}(Ry))^{\frac{1}{n}} - 1)^n |dy\right)^{\frac{1}{n}} \le \frac{1}{R}\left(\int_{B_{CR}} |f(z)^{\frac{1}{n}} - 1|^n dz\right)^{\frac{1}{n}}.$$

By the assumption (FA) the integral is finite, thus we have proved that

$$-\min_{\bar{O}}(\xi - \bar{\xi}) \le C/R, \quad x \in O$$

Similarly we also have $-\min_{\bar{O}}(\bar{\xi} - \xi) \le C/R$. (5.3) is proved.

Next we set

$$E_M := \{x; \quad (x - \bar{x})' D^2\xi(\bar{x})(x - \bar{x}) \le 1\}$$

where $\bar{x}$ is the minimum of $\bar{\xi}$. By Theorem 1 of [5] $\bar{x}$ is the unique minimum point of $\bar{\xi}$. Then by the same argument as in [9] we have the following: There exist $\bar{k}$ and C depending only on n and f such that for $\epsilon = \frac{1}{10}$, $M = 2^{(1+\epsilon)k}$, $2^{k-1} \le M' \le 2^k$,

$$\left(\frac{2M'}{R^2} - C2^{-\frac{3\epsilon k}{2}}\right)^{\frac{1}{2}} E_M \subset \frac{1}{R} a_M(\Omega_{M'}) \subset \left(\frac{2M'}{R^2} + C2^{-\frac{3\epsilon k}{2}}\right)^{\frac{1}{2}} E_M, \quad \forall k \ge \bar{k},$$

which is

$$\sqrt{2M'}\left(1 - \frac{C}{2^{\epsilon k/2}}\right) E_M \subset a_M(\Omega_{M'}) \subset \sqrt{2M'}\left(1 + \frac{C}{2^{\epsilon k/2}}\right) E_M.$$

Let Q be a positive definite matrix such that $Q^2 = D^2\bar{\xi}(\bar{x})$, O be an orthogonal matrix such that $T_k := OQ_k a_M$ is upper triangular. Then clearly $det(T_k) = 1$ and by Proposition 3.4 of [9] we have

$$\|T_k - T\| \le C2^{-\frac{\epsilon k}{2}}.$$

Let $v = u \cdot T$, then clearly

$$det(D^2 v(x)) = f(Tx).$$

For v and some $\bar{k}$ large we have

$$\sqrt{2M'}\left(1 - \frac{C}{2^{\epsilon k/2}}\right) B \subset \{x; \quad v(x) < M'\} \subset \sqrt{2M'}\left(1 + \frac{C}{2^{\epsilon k/2}}\right) B \quad \forall M' \ge 2^{\bar{k}}.$$

Consequently

$$\left|v(x) - \frac{1}{2}|x|^2\right| \le C|x|^{2-\epsilon}. \tag{5.4}$$

Clearly $f(T\cdot)$ also satisfies (FA). Proposition 2.1 gives the asymptotic behavior of u and the estimates on its derivatives. the constant d in the estimate in two dimensional spaces is determined similarly as in [9]. Theorem 1.2 is established. □

Remark 5.1 Corollary 1.1 follows from Theorem 1.2 just like in [9] so we omit the proof.

Appendix: Interior estimate of Caffarelli and Jian–Wang

The following theorem is a combination of the interior estimate of Caffarelli [7] and an improvement by Jian and Wang [25].

Theorem 6.1 (Caffarelli, Jian and Wang) *Let $u \in C^0(\Omega)$ be a convex viscosity solution of*

$$\begin{aligned} &det(D^2u) = f, \quad \Omega, \\ &u = 0 \quad on \quad \partial\Omega, \end{aligned}$$

where Ω is a convex bounded domain satisfying $B_1 \subset \Omega \subset B_n$. Assume that f is Dini continuous on Ω and

$$\frac{1}{c_0} \le f \le c_0, \quad \Omega.$$

Then $u \in C^2(B_{1/2})$ and $\forall x, y \in B_{1/2}$

$$|D^2u(x) - D^2u(y)| \le C\left(d + \int_0^d \frac{\omega_f(r)}{r} + d\int_d^1 \frac{\omega_f(r)}{r^2}\right) \tag{6.1}$$

where $d = |x-y|$, $C > 0$ depends only on n and c_0, ω_f is the oscillation function of f defined by

$$\omega_f(r) := \sup\{|f(x) - f(y)| : \quad |x-y| \le r\}.$$

It follows that (i) If f is Dini continuous, then $u \in C^2(B_{1/2})$, and the modulus of convexity of D^2u can be estimated by (6.1). (ii) If $f \in C^\alpha(\Omega)$ and $\alpha \in (0,1)$, then

$$\|D^2u\|_{C^\alpha(B_{1/2})} \le C\left(1 + \frac{\|f\|_{C^\alpha(\Omega)}}{\alpha(1-\alpha)}\right).$$

(iii) If $f \in C^{0,1}(\Omega)$, then

$$|D^2u(x) - D^2u(y)| \le Cd(1 + \|f\|_{C^{0,1}(\Omega)}|\log d|).$$

Here we recall that f is Dini continuous if the oscillation function ω_f satisfies $\int_0^1 \omega_f(r)/r dr < \infty$.

Remark 6.1 Note that in Caffarelli's interior estimate $u = 0$ is assumed on $\partial\Omega$. Since Ω is very close to a ball, by [5,6] u is strictly convex in Ω. But there is no explicit formula that describes how the higher order derivatives of u depend on f. In Jian-Wang's theorem, this dependence is given as in (6.1) but instead of assume $u = 0$ on $\partial\Omega$, they assumed u is strictly convex and their constant depends on the strict convexity. We feel the way that Theorem 6.1 is stated is more convenience for application. We only used the (ii) and (iii) of Theorem 6.1 in this article.

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