Clarkson-Erdös-Schwartz Theorem on a Sector

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Abstract The analogues results of the Clarkson-Erdös-Schwartz Theorem on a closed sector are obtained, i.e., some sufficient conditions are obtained for the incompleteness and minimality of the Müntz system $E(\Lambda)$ in H_α and each element in the closure of the linear span of Müntz system $E(\Lambda)$ can be extended analytically throughout $\text{int}(I_\pi) = \{z : |z| < 1, |\arg z| < \pi\}$ with a series expansion of the form $\sum a_k z^{\lambda_k}$, where H_α is a Banach space consisting of all complex continuous functions f on the closed sector $I_\alpha = \{z : |z| \leqslant 1, |\arg z| \leqslant \alpha\}$ $(0 \leqslant \alpha < \pi)$, analytic in the interior of I_α, and the norm is given by $||f|| = \max\{|f(z)| : z \in I_\alpha\}$.

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1 Introduction

Suppose $\Lambda = \{\lambda_n : n = 1, 2, \cdots\}$ is a sequence of positive real numbers arranged for convenience in non-decreasing order:

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \quad \text{and } \delta(\Lambda) = \inf\{\lambda_{n+1} - \lambda_n : n \geqslant 0\}.$$

$E(\Lambda) = \{z^{\lambda_n} = \exp\{\lambda_n \log z\} : n = 0, 1, 2, \cdots\}$ is called the Müntz system, span$E(\Lambda)$ is the linear span of the Müntz system. The elements of the set span$E(\Lambda)$ are called the Müntz polynomial or the $\Lambda-$polynomial ([4]). Let K be a compact set in the plane, and let $C(K)$ be the space of all continuous functions on K, equipped with the uniform norm. The famous Müntz theorem ([4] and [18]) states that span$E(\Lambda)$ is dense in $C([0, 1])$ if and only if $\sum_{n=1}^{\infty} 1/\lambda_n$ diverges. Moreover, If the set span$E(\Lambda)$ is not a dense subspace of $C[0, 1]$, it is natural to ask what is its topological closure, this problem is first solved by Clarkson and Erdös ([5]) for the case of integer exponents Λ, they proved that $\sum_{n=1}^{\infty} 1/\lambda_n$ converges implies that the elements in the closure of span$E(\Lambda)$ can be extended analytically throughout to the unit disc with a series expansion of the form

$$f(x) = \sum_{k=0}^{\infty} a_k x^{\lambda_k}, \ 0 \leqslant x < 1. \qquad (1)$$

this same question was tackled by L.Schwarts ([16])for certain strictly increasing sequences of exponents (he assumed $\delta(\Lambda) > 0$ and by Borwein ([4]) and Erdĺyi ([6]). Now, these results are called Clarkson-Erdös-Schwartz Theorem([1]). On the other hand, if K is a compact in the plane whose complement is connected, the Mergelyan theorem ([15]) asserts that the set of all polynomials is dense in $C(K)$. It is a nontrivial problem to establish a Clarkson-Erdös-Schwartz

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Theorem on a closed sector in the plane. The aim of this paper is to give an answer to this nontrivial problem. First we introduce some notations and definitions. Let B be a Banach space. If $E = \{e_k : k = 1, 2, \cdots\} \subset B$, let span$E$ denote the subspace of B, consisting of all finite linear combinations of E and let $\overline{\text{span}}E$ be the closure of span$E$ in B. The set E is said to be *incomplete* in B ([17]) if $\overline{\text{span}}E$ does not coincide with the whole B. The set E is called to be *minimal* in B ([17]) if no element of E belongs to the closure of the vector subspace generated by the other elements of E, i.e., for all $e \in E$, $e \notin \overline{\text{span}}(E - \{e\})$. The minimality of the E is equivalent to the existence of $\{f_k : k = 1, 2, \cdots\}$ of conjugate functionals in the dual Banach space B^* of B; the latter means that $f_n(e_m) = \delta_{nm}$ (Kronneker delta, i.e., $\delta_{nn} = 1$, while $\delta_{nm} = 0$ for $n \neq m$). $\{f_n : n = 1, 2, \cdots\}$ is also called a biorthogonal system of E. It follows that if E is minimal, each $x \in \overline{\text{span}}E$ has a unique formal $E-$expansion $\sum x_n e_n$ ([17]), where $x_n = f_n(x)$.

In this paper, we particularize B to be the Banach space H_α consisting of all functions $f(z)$ which are continuous on the closed sector $I_\alpha = \{z = re^{i\theta} : 0 \leqslant r \leqslant 1, |\theta| \leqslant \alpha\}$ $(0 \leqslant \alpha < \pi)$, analytic in int$(I_\alpha)$, the interior of I_α, the norm of f is given by

$$||f|| = \max\{|f(z)| : z \in I_\alpha\}.$$

If the Banach space H_α is replaced by the Fréchlet space F_α, which consists of all functions analytic in the sector intI_α, under the compact topology (uniform convergence on each compact subset of intI_α), Khabibullin ([8]), Rubel ([14]) and Malliavin ([13]) have proved that span$E(\Lambda)$ is not dense in F_α if and only if there are $b \in (0, \frac{\pi}{\alpha})$ and M_b such that

$$\lambda(y) - \lambda(x) \leqslant b \log y - b \log x + M_b \quad (y > x \geqslant 1),$$

where the characteristic logarithm $\lambda(t)$ is defined by ([13] and [14])

$$\lambda(t) = \sum_{0<\lambda_n \leqslant t} \lambda_n^{-1}. \tag{2}$$

Inspired by the method of Khabibullin ([8]), Anderson ([2]), Rubel and Malliavin, we obtain the following Theorem.

Theorem . *Let $\alpha \in [0, \pi)$ and $\Lambda = \{\lambda_n : n = 1, 2, \cdots\}$ be a sequence satisfying $\delta(\Lambda) > 0$ and there exists a decreasing function $\varepsilon(x)$ on $[0, \infty)$ such that $\varepsilon(x) \to 0$ as $x \to \infty$ and such that*

$$\lambda(y) - \lambda(x) \leqslant \frac{\alpha}{\pi} \log y - \frac{\alpha}{\pi} \log x + \varepsilon(x), \quad (y > x \geqslant 1) \tag{3}$$

Then $E(\Lambda)$ is minimal and $\overline{span}E(\Lambda)$ does not contain the function z^λ for $\lambda \notin \Lambda$, $Re\lambda > 0$, and each function f in $\overline{span}(E(\Lambda))$ can be extended analytically throughout the region int(I_π) with a series expansion of the form (1).

Remark 1 If $\lambda(t)$ is bounded on $[1, \infty)$, then the function $\varepsilon(x) = \lambda(\infty) - \lambda(x)$ is decreasing, $\varepsilon(x) \to 0$ as $x \to \infty$ and satisfies (3) with $\alpha = 0$. So we have the following Corollary which can be found in [4] (p.178).

Corollary *Let $\Lambda = \{\lambda_n : n = 1, 2, \cdots\}$ be a sequence satisfying $\delta(\Lambda) > 0$. If $\lambda(t)$ is bounded on $[1, \infty)$, then $E(\Lambda) \subset H_0$ is minimal and each function f in $\overline{span}(E(\Lambda))$ can be extended analytically throughout the region int(I_π) with a series expansion of the form* (1). *If $\Lambda = \{\lambda_n :$*

$n = 1, 2, \cdots\}$ *is a sequence of distinct positive integers and* $\lambda(t)$ *is bounded on* $[1, \infty)$, *then each function* f *in* $\overline{span}(E(\Lambda))$ *can be extended analytically throughout the open unit disk.*

Therefore, Theorem is a generalization of the Clarkson-Erdös-Schwartz Theorem to a sector.

2 Lemmas and Proofs

In order to prove Theorem, we need the following technical lemmas. The following Lemma 1 can be seen from [3] and [12].

Lemma 1 (Fuchs' Lemma). *If* Λ *is a sequence of positive numbers satisfying* $\delta(\Lambda) > 0$, *then the function*

$$G(z) = \prod_{n=1}^{\infty} \left(\frac{\lambda_n - z}{\lambda_n + z} \right) \exp\left(\frac{2z}{\lambda_n} \right) \tag{4}$$

is a meromorphic function and satisfies the following inequalities:

$$|G(z)| \leq \exp\{x\lambda(|z|) + Ax\}, \quad z \in \mathbb{C}_+, x \geqslant 0, \tag{5}$$

$$|G(z)| \geq \exp\{x\lambda(|z|) - Ax\}, \quad z \in C(\Lambda, \delta_0), \tag{6}$$

where $4\delta_0 = \delta(\Lambda)$ *and*

$$C(\Lambda, \delta_0) = \{z \in \mathbb{C}_+ : |z - \lambda_n| \geqslant \delta_0, n = 1, 2, \cdots\}. \tag{7}$$

Lemma 2. *Let* $\varepsilon(x)$ *be a positive decreasing function on* $[0, \infty)$, $\Lambda = \{\lambda_n : n = 1, 2, \cdots\}$ *and* $\Lambda' = \{\lambda'_n : n = 1, 2, \cdots\}$ *sequences of positive numbers satisfying* $\delta(\Lambda) > 0$ *and* $\delta(\Lambda') > 0$, *respectively. If*

$$\lambda(y) - \lambda(x) \leqslant \lambda'(y) - \lambda'(x) + \varepsilon(x), \ y > x \geq 0, \tag{8}$$

then there exist a constant A_1 *and a subsequence* $\Lambda^* = \{\lambda^*_n : n = 1, 2, \cdots\}$ *of the sequence* $\Lambda' = \{\lambda'_n : n = 1, 2, \cdots\}$ *such that*

$$|\lambda(x) + \lambda^*(x) - \lambda'(x) - A_1| \leqslant \varepsilon(x) + x^{-1}, \ x > 0. \tag{9}$$

Proof of Lemma 2. Similar to the proof in [13, p.181-182] and [14, p.148-149], we define

$$\varphi(x) = \inf\{\lambda'(s) - \lambda(s) : s \geqslant x\}.$$

It follows from (8) that $\varphi(x) \geqslant \lambda'(x) - \lambda(x) - \varepsilon(x)$.

Now $\varphi(x)$ is constant except for possible jumps at the jumps of $\lambda'(x)$. Let a be a point of discontinuity of φ. Then, the left limit of φ at a is $\varphi(a - 0) = \lambda'(a-) - \lambda(a - 0)$ and the right limit of φ at a is $\varphi(a + 0) = \varphi(a) \leqslant \lambda'(a) - \lambda(a)$. We denote by $\Delta\varphi(a) = \varphi(a + 0) - \varphi(a - 0)$, the jump of φ at a. Then

$$\Delta\varphi(a) \leqslant \Delta\lambda'(a) - \Delta\lambda(a) \leqslant \Delta\lambda'(a). \tag{10}$$

Therefore, there exists a sequence $\Lambda^* = \{\lambda_n^* : n = 1, 2, \cdots\}$ of positive numbers whose counting function ([13] and [14]) is $\Lambda^*(t) = [\Phi(t)]$, where $[x]$ denotes the integral part of x,

$$\Phi(t) = \int_0^t s \; d\varphi(s) \;\; \text{and } \Lambda^*(t) = \sum_{\lambda_n^* \leqslant t} 1.$$

The characteristic logarithm $\lambda^*(t)$ of the sequence Λ^* is constant except possibly the jumps of $\varphi(t)$, and we have $\Delta\lambda^*(a) < a^{-1} + \Delta\varphi(a)$. Using (10), we get $\Delta\lambda^*(a) < a^{-1} + \Delta\lambda'(a)$. Furthermore, $a\Delta\lambda^*(a)$ and $a\Delta\lambda'(a)$ must be integers, so $\Delta\lambda^*(a) \leqslant \Delta\lambda'(a)$ and this means that Λ^* is a subsequence of Λ'. Now,

$$\varphi(x) - \varphi(0) = \int_0^x s^{-1} \; d\Phi(s) \quad \text{and} \quad \lambda^*(x) = \int_0^x s^{-1} \; d[\Phi(s)].$$

An integration by parts shows that

$$\lambda^*(x) - \varphi(x) = A_1 + \varepsilon_2(x),$$

where

$$\varepsilon_2(x) = \int_x^\infty (\Phi(s) - [\Phi(s)])\frac{ds}{s^2} - x^{-1}(\Phi(x) - [\Phi(x)])$$

and

$$A_1 = \int_0^\infty (\Phi(s) - [\Phi(s)])\frac{ds}{s^2} - \varphi(0).$$

We now define

$$\varepsilon_1(x) = \lambda(x) + \lambda^*(x) - \lambda'(x) - A_1,$$

It is clear that $|\varepsilon_2(x)| \leqslant x^{-1}$, so by the definition of $\varphi(x)$, $\varepsilon_1(x) \leqslant \varepsilon_2(x) \leqslant x^{-1}$. (8) is simply another way of saying that

$$\varepsilon_1(x) \geqslant \varepsilon_2(x) - \varepsilon(x) \geqslant -x^{-1} - \varepsilon(x).$$

This proves (9).

Lemma 3. *Let $b \geqslant 0$ and let $\Lambda = \{\lambda_n : n = 1, 2, \cdots\}$ be a sequences of positive real numbers satisfying $\delta(\Lambda) > 0$. If there exists a constant A_2 such that*

$$\lim_{x\to\infty} |\lambda(x) - b\log^+ x - A_2| = 0, \tag{11}$$

then the function

$$g_0(z) = \frac{G(z)e^{-a_0 z}}{\Gamma(\frac{1}{2} + 2bz)}, \tag{12}$$

is meromorphic and satisfies

$$\limsup_{x\to\infty} x^{-1} \log|g_0(x)| = 0, \tag{13}$$

$$\lim_{x\in C(\Lambda,\delta_0), x\to\infty} x^{-1} \log|g_0(x)| = 0, \tag{14}$$

and

$$\lim_{k\to\infty} \lambda_k^{-1} \log|g_0'(\lambda_k)| = 0, \tag{15}$$

where $\Gamma(z)$ is the Euler Gamma function, $G(z)$ is defined by (4) , $a_0 = 2A_2 - 2b\log(2b)$ ($a_0 = 2A_2$, if $b = 0$) and $C(\Lambda, \delta_0)$ is defined by (7) .

Proof of Lemma 3. The main method of the proof is based on the use of the following function used by Malliavin ([11])

$$\psi(s) = 2 + s\log\left|\frac{s-1}{s+1}\right|.$$

The function $\psi(s)$ is decreasing on $[0,1)$, increasing on $(1,\infty)$ and there exists $s_0 \in (\frac{5}{6},\frac{6}{7})$ such that $\psi(s_0)=0$. Thus $\psi(s)$ is negative on $(s_0,1)\bigcup(1,\infty)$. Since $\delta(\Lambda)>0$, $\sum_{n=1}^{\infty}|\lambda_n|^{-2}$ converges. Thus $G(z)$ defined by (4) is the quotient of convergent canonical products. As a result, the product (4) defines a meromorphic function in the complex plane $\mathbb{C}$, which has zeros at each point λ_n. Writing $\log|G(x)|$ as a sum of logarithms, and that sum as a Stieljes integral, we get

$$\log|G(x)| = x\int_0^{\infty}\psi\left(\frac{t}{x}\right)d\lambda(t).$$

Let

$$k(x) = \lambda(x) - b\log^{+}x - A_2, \quad \varepsilon(x) = \sup\{|k(y)| : y \geqslant x\}$$

and

$$\varepsilon_3(x) = -\int_0^{x}\log\left|\frac{1-t}{1+t}\right|dt.$$

Then the function $\varepsilon_3(x)$ is continuous on $[0,\infty)$, increasing and positive on $(0,\infty)$, convex on $[0,1]$ and concave on $[1,\infty)$. Thus $x^{-1}\varepsilon_3(x)$ is increasing on $(0,1]$ and decreasing on $[1,\infty)$, so $\sup\{x^{-1}\varepsilon_3(x) : x>0\} = \varepsilon_3(1) = 2\log 2 < 3$. An easy calculation shows that

$$\int_0^{\infty}\psi\left(\frac{t}{x}\right)d\log^{+}t = \int_{1/x}^{+\infty}\psi(t)\frac{dt}{t} = 2\log x - 2 + \varepsilon_3(x^{-1}),$$

and the Gamma function $\Gamma(z)$ satisfies

$$\log\left|\Gamma(\frac{1}{2}+z)\right| = x\log\left|z+\frac{1}{2}\right| - \left|y\arg(z+\frac{1}{2})\right| - x + c_1(z), \tag{16}$$

where $c_1(z)$ satisfies $|c_1(z)| \leqslant 10$ for $x = \mathrm{Re}z \geqslant 0$. By the choice of a_0, we obtain

$$x^{-1}\log|g_0(x)| = I_1(x) - 2A_2 + \int_0^{\infty}\psi\left(\frac{t}{x}\right)dk(t), \tag{17}$$

where function $I_1(x) = b\varepsilon_3(x^{-1}) - x^{-1}c_1(2bx)$ satisfies

$$\lim_{x\to\infty}|I_1(x)| = 0.$$

Since $k(x)$ has a jump at each point λ_n and $\varepsilon(x)\to 0$ as $x\to\infty$, we can assume, without loss of generality, that $1 \geqslant 6\varepsilon(x) > 0$ for $x \geq 0$(if not, replaced $\varepsilon(x)$ by $\min\{\frac{1}{6},\varepsilon(x)\}$). Let $a(x) = 1+\varepsilon(\frac{6x}{7})$. If we split the range of the integral in (17) into the ranges $(0,\frac{x}{a(x)}]$, $[\frac{x}{a(x)}, xa(x)]$ and $[xa(x),\infty)$, integration by parts in $(0,\frac{x}{a(x)}]$ and $[xa(x),\infty)$, respectively, then $x^{-1}\log|g_0(x)|$ can be written in the form

$$x^{-1}\log|g_0(x)| = \sum_{j=1}^{8}I_j(x),$$

where

$$I_2(x) = k\left(\frac{x}{a(x)}\right)\psi\left(\frac{1}{a(x)}\right) - k(xa(x))\psi(a(x));$$

$$I_3(x) = -\int_0^{\frac{6x}{7}} k(t)\psi'\left(\frac{t}{x}\right)\frac{dt}{x};\quad I_4(x) = -\int_{\frac{7x}{6}}^{\infty} k(t)\psi'\left(\frac{t}{x}\right)\frac{dt}{x};$$

$$I_5(x) = -\int_{\frac{6x}{7}}^{\frac{x}{a(x)}} k(t)\psi'\left(\frac{t}{x}\right)\frac{dt}{x};\quad I_6(x) = -\int_{xa(x)}^{\frac{7x}{6}} k(t)\psi'\left(\frac{t}{x}\right)\frac{dt}{x};$$

$$I_7(x) = -b\int_{\frac{x}{a(x)}}^{xa(x)} \psi\left(\frac{t}{x}\right)d\log^+ t \text{ and } I_8(x) = \int_{\frac{x}{a(x)}}^{xa(x)} \psi\left(\frac{t}{x}\right)d\lambda(t).$$

Next we shall show that

$$\lim_{x\to\infty}|I_j(x)| = 0, j = 2, 3, \cdots, 7. \tag{18}$$

Since $1 < a(x) = 1 + \varepsilon\left(\frac{6x}{7}\right) \leqslant \frac{7}{6}$,

$$0 \leqslant -\psi\left(\frac{t}{x}\right) \leqslant -\log\varepsilon\left(\frac{6x}{7}\right)\ \left(t \in \left[\frac{6x}{7}, \frac{x}{a(x)}\right]\right)$$

and

$$0 \leqslant -\psi\left(\frac{t}{x}\right) \leqslant -2\log\varepsilon\left(\frac{6x}{7}\right)\quad (t \in [xa(x), \infty)\ ),$$

we see that

$$|I_2(x)| \leqslant -3\varepsilon\left(\frac{6x}{7}\right)\log\varepsilon\left(\frac{6x}{7}\right);$$

$$|I_4(x)| \leqslant -\varepsilon(x)\int_{\frac{7x}{6}}^{\infty}\psi'\left(\frac{t}{x}\right)\frac{dt}{x} = -\varepsilon(x)\psi\left(\frac{7}{6}\right);$$

$$|I_5(x)| \leqslant -\varepsilon\left(\frac{6x}{7}\right)\int_{\frac{6x}{7}}^{\frac{x}{a(x)}}\psi'\left(\frac{t}{x}\right)\frac{dt}{x} \leqslant -2\varepsilon\left(\frac{6x}{7}\right)\log\varepsilon\left(\frac{6x}{7}\right);$$

$$|I_6(x)| \leqslant \varepsilon\left(\frac{6x}{7}\right)\int_{xa(x)}^{\frac{6x}{7})}\psi'\left(\frac{t}{x}\right)\frac{dt}{x} \leqslant -2\varepsilon\left(\frac{6x}{7}\right)\log\varepsilon\left(\frac{6x}{7}\right).$$

These prove that (18) hold for $j = 2, 4, 5, 6$. Also for $x > 1$,

$$|I_3(x)| \leqslant -\varepsilon(0)\int_0^{\frac{\sqrt{x}}{2}}\psi'\left(\frac{t}{x}\right)\frac{dt}{x} - \varepsilon\left(\frac{\sqrt{x}}{2}\right)\int_{\frac{\sqrt{x}}{2}}^{\frac{6x}{7}}\psi'\left(\frac{t}{x}\right)\frac{dt}{x}$$

$$\leqslant \frac{\varepsilon(0)}{\sqrt{x}} + 2\varepsilon\left(\frac{\sqrt{x}}{2}\right),$$

so (18) also holds for $j = 3$. Since $\frac{1}{a(x)} \geqslant \frac{6}{7} > s_0$, so

$$0 \leqslant I_7(x) \leqslant b\varepsilon\left(\frac{6x}{7}\right)\int_{\frac{x}{a(x)}}^{xa(x)}\left(-\frac{t}{x}\log\left|\frac{t}{x} - 1\right|\right)d\log^+ t$$

and

$$0 \leqslant -I_8(x) \leqslant \int_{\frac{x}{a(x)}}^{xa(x)}\left(-\frac{t}{x}\log\left|\frac{t}{x} - 1\right|\right)d\lambda(t).$$

Therefore

$$0 \leqslant I_7(x) \leqslant 2b\varepsilon\left(\frac{6x}{7}\right)\int_0^{\varepsilon(\frac{6x}{7})}(-\log s)ds \leqslant -4b\left(\varepsilon\left(\frac{6x}{7}\right)\right)^2\log\varepsilon\left(\frac{6x}{7}\right).$$

Hence (18) holds for $j=7$. Finally,

$$0 \leqslant -I_8(x) \leqslant x^{-1} \sum_{\frac{x}{a(x)}<\lambda_n \leqslant xa(x)} \left(\log(3x) - \log|\lambda_n - x|\right).$$

let $\Lambda(t)=\sum_{\lambda_n\leqslant t} 1$ be the counting function of Λ ([3]), then for $x\in C(\Lambda,\delta_0)$, we have $|\lambda_n - x| \geqslant |n-\Lambda(x)|\delta_0$. Let

$$n_1(x) = \max\left\{\Lambda(xa(x)) - \Lambda(x), \Lambda(x) - \Lambda\left(\frac{x}{a(x)}\right)\right\},$$

then, for $x\in C(\Lambda,\delta_0)$,

$$-I_8(x) \leqslant \frac{2}{x}\left(n_1(x)\log\left(\frac{3x}{\delta_0}\right) - \log n_1(x)!\right).$$

By $e^n n! \geqslant n^n (n\geqslant 1)$,

$$-I_8(x) \leqslant \frac{2}{x} n_1(x)\left(\log\left(\frac{3ex}{\delta_0}\right) - \log n_1(x)\right).$$

Since the inequalities

$$\Lambda(R) - \Lambda(r) \leqslant R(\lambda(R) - \lambda(r)) \leqslant 2R\varepsilon(r) + bR\log\frac{R}{r}$$

hold for $R>r$, we see that

$$n_1(x) \leqslant 2xa(x)\varepsilon\left(\frac{6x}{7}\right) + bxa(x)\log a(x) \leqslant 4x(1+b)\varepsilon\left(\frac{6x}{7}\right).$$

The function $t(\log a - \log t)$ is increasing on $(0, ae^{-1})(a>0)$ and there is $x_0>1$ such that $9\delta_0(1+b)\varepsilon\left(\frac{6x}{7}\right)\leqslant 3$, we see that

$$-I_8(x) \leqslant -18(1+b)\varepsilon\left(\frac{6x}{7}\right)\log\left(\delta_0(1+b)\varepsilon\left(\frac{6x}{7}\right)\right), \quad x\geq x_0.$$

These prove that (13) and (14) hold. Similarly, (15) can also be proved. This completes the proof of Lemma 3.

3 Proof of Theorem

Proof . We can assume that $\alpha>0$ in the proof of Theorem. It is a consequence of the Hahn-Banach theorem ([15]) that $\overline{\mathrm{span}}E(\Lambda)\neq H_\alpha$ if and only if there exists a bounded linear functional T on H_α with $||T||=1$ which vanishes on all of $E(\Lambda)$. Since every bounded linear functional on H_α is given by integration with respect a complex Borel measure on I_α. So we shall construct a bounded linear functional T on H_α such that

$$T(\zeta^z) = g(z) = \frac{z^2 G(z) e^{-Az}}{\Gamma(\frac{1}{2}+\frac{2}{\pi}\alpha z)(1+z)^4},$$

where $\Gamma(z)$ is the Euler Gamma function, $G(z)$ is defined by (4) and A is a sufficient large positive constant. The function $g(z)$ is analytic in the right half plane $\mathbb{C}_+$. Moreover, since

$G(z)G(-z) \equiv 1$ and $\Gamma(z)\Gamma(1-z)\sin(\pi z) \equiv \pi$, it follows from Lemma 3, (16) and Cauchy's formula for $g'(z)$ and $g''(z)$ that

$$|g(z)| + |g'(z)| + |g''(z)| \leqslant \frac{Ae^{\alpha|y|}}{1+|z|^2} \quad (x \geqslant 0) \tag{19}$$

holds for a sufficient large positive constant A. Fix z so that $x > 0, y > 0$, and consider the Cauchy's formula for $g(z)e^{\alpha zi}$, where the path of integration consists of the quadrant circle with center at 0, radius $R > 1 + |z|$ from R to iR, followed by the interval from iR to 0 and by the interval from 0 to R. The integral over the quadrant circle tends to 0 as $R \to \infty$, so we are left with

$$g(z)e^{i\alpha z} = \frac{1}{2\pi i}\int_0^{+\infty} \frac{g(t)e^{i\alpha t}}{t-z}dt - \frac{i}{2\pi i}\int_0^{+\infty} \frac{g(it)e^{i\alpha it}}{it-z}dt \tag{20}$$

and similarly, fix z so that $x > 0, y > 0$, and consider the Cauchy formula for $g(z)e^{-i\alpha z}$, where the path of integration consists of the lower quadrant circle with center at 0, radius $R > 1+|z|$ from $-iR$ to R, followed by the interval from R to 0 and by the interval from 0 to $-iR$. The integral over the lower quadrant circle tends to 0 as $R \to \infty$, so we are left with

$$0 = \frac{-1}{2\pi i}\int_0^{+\infty} \frac{g(t)e^{-i\alpha t}}{t-z}dt + \frac{i}{2\pi i}\int_0^{-\infty} \frac{g(it)e^{-i\alpha it}}{it-z}dt. \tag{21}$$

Using

$$\frac{1}{z-it} = \int_0^1 s^{z-it-1}ds \quad \text{and} \quad \int_{-\alpha}^{\alpha} e^{i\theta(t-z)} i d\theta = \frac{e^{i\alpha(it-z)} - e^{-i\alpha(it-z)}}{t-z}$$

(20) multiplied by $e^{-\alpha zi}$ plus (21) multiplied by $e^{\alpha zi}$, we obtain, for $z = x+iy, x > 0, y > 0$,

$$g(z) = \frac{1}{2\pi}\int_0^{+\infty} g(t)\int_{-\alpha}^{\alpha} e^{i\theta(t-z)}d\theta dt \tag{22}$$

$$-\frac{1}{2\pi}\int_{-\infty}^{0} g(it)\int_0^1 (se^{i\alpha})^{(z-it)}\frac{ds}{s}dt - \frac{1}{2\pi}\int_0^{+\infty} g(it)\int_0^1 (se^{-i\alpha})^{(z-it)}\frac{ds}{s}dt.$$

Similarly, (22) also holds for $x > 0, y < 0$. The interchange in order of integration in (22) are legitimate: in the integrants in (22) are replaced by their absolute values, some finite integral results. Hence (22) can be rewritten in the form

$$g(z) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} e^{i\theta z}h_0(e^{i\theta})d\theta$$

$$+\frac{1}{2\pi i}\int_0^1 ((se^{-i\alpha})^z h_1(se^{-i\alpha}) - (se^{i\alpha})^z h_{-1}(se^{i\alpha}))\frac{ds}{s},$$

where

$$h_l(\zeta) = \int_{L_l} g(z)\zeta^{-z}dz \tag{23}$$

and $L_l = \{t\exp\{\frac{\pi}{2}li\} : t \geqslant 0\}$ $(l \in \{-1,0,1\})$ are half-lines. By (21), $h_0(\zeta)$ is analytic in the region $D_0 = \{\zeta : |\zeta| > 1, |\arg\zeta| < \pi\}$ and continuous in the set $\overline{D_0} = \{\zeta : |\zeta| \geqslant 1, |\arg\zeta| < \pi\}$, each function $h_l(\zeta)(l = \pm 1)$ is analytic in the sector $D_l = \{\zeta : \alpha < -l\arg\zeta < \pi\}$ and continuous in the closure $\overline{D_l} = \{\zeta : \alpha \leqslant -l\arg\zeta \leqslant \pi\}$ of D_l. By Cauchy's formula, $h_0(\zeta)$ can be continued analytically to a bounded analytic function in the region $D_{-1}\bigcup D_0\bigcup D_1$ $= \{\zeta = \rho e^{i\phi} : \zeta \notin I_\alpha, |\phi| < \pi\}$, i.e., $h_0(\rho e^{i\phi}) = h_l(\rho e^{i\phi})$ for $\rho > 1, \alpha < -l\phi < \pi, l = \pm 1$. By

(21), $h_0(\zeta)$ is bounded in the circular arc $\{\zeta : |\zeta| = 1, |\arg\zeta| < \alpha\}$. Integrations by parts twice in (23),

$$h_l(se^{-il\alpha}) = (\log s - il\alpha)^{-2}\int_{L_l} g''(z)(se^{-il\alpha})^{-z}dz, \quad l = \pm 1.$$

By (19),

$$\int_0^1 (|h_{-1}(se^{i\alpha})| + |h_1(se^{-i\alpha})|)\frac{ds}{s} < \infty.$$

Therefore, the linear functional

$$T(\varphi) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \varphi(e^{i\theta})h_0(e^{i\theta})d\theta$$

$$+\frac{1}{2\pi i}\int_0^1 (\varphi(se^{-i\alpha})h_1(se^{-i\alpha}) - \varphi(se^{i\alpha})h_{-1}(se^{i\alpha}))\frac{ds}{s}$$

is a bounded linear functional on H_α and satisfies $T(\zeta^\lambda) = g(\lambda)$ for $\lambda \in \mathbb{C}_+$. By the Riesz representation theorem, $z^\lambda \notin \overline{\text{span}}E(\Lambda)$ for $\lambda \notin \Lambda$ and $\text{Re}\lambda > 0$. Similarly, replacing $g(z)$ by $(z-\lambda)^{-1}g(z)$ for $\lambda \in \Lambda$, we can also prove that no element of $E(\Lambda)$ belongs to the closure of the vector subspace generated by the other elements of $E(\Lambda)$. Therefore, $E(\Lambda)$ is minimal, and $\overline{\text{span}}M(\Lambda) \neq H_\alpha$.

Next, define, for $0 < b = \frac{\alpha}{\pi} < \infty$, the arithmetic progression ([13]) Λ_b by

$$\Lambda_b = \left\{\frac{n}{b} : n = 1, 2, \cdots\right\}$$

and observe that the counting function $\Lambda_b(t) = \sum_{n \leqslant b} 1$ of Λ_b satisfies $\Lambda_b(t) = [bt] = bt + O(1)$, and the characteristic logarithm $\lambda_b(t)$ of the sequence Λ_b satisfies

$$\lambda_b(t) = b\log t + b\log b + b\gamma + O(t^{-1}),$$

as $t \to \infty$, where γ is a Euler constant. So by Lemma 2, there exist a constant A_1 and a subsequence $\Lambda^* = \{\lambda_n^* : n = 1, 2, \cdots\}$ of Λ_b such that (9) holds. If Λ and Λ^* have common elements or $\delta(\Lambda \cup \Lambda^*) = 0$, we adjust Λ^* as follows: let $4h_1 = min\{\delta(\Lambda), b\}$ and $n_k \in \mathbb{N}$ such that $\lambda_{n_k} \leqslant \lambda_k^* \leqslant \lambda_{1+n_k}$ and let

$$\lambda_k^{**} = \begin{cases} \lambda_k^*, & \text{if } \lambda_{n_k} + h_1 \leqslant \lambda_k^* < \lambda_{1+n_k} - h_1; \\ \lambda_k^* + h_1, & \text{if } \lambda_{n_k} \leqslant \lambda_k^* < \lambda_{n_k} + h_1; \\ \lambda_k^* - h_1, & \text{if } \lambda_{1+n_k} - h_1 < \lambda_k^* < \lambda_{1+n_k}, \end{cases}$$

and let

$$A_3 = \sum_{k=1}^{+\infty}\left(\frac{1}{\lambda_k^*} - \frac{1}{\lambda_k^{**}}\right),$$

then the set $\Lambda^{**} = \{\lambda_n^{**} : n = 1, 2, \cdots\}$ and the set Λ are disjoint and $\delta(\Lambda \cup \Lambda^{**}) \geqslant h_1 > 0$. For $x \geqslant 2h_1 + 1$, we have the following inequalities:

$$|\lambda^*(x) - \lambda^{**}(x) - A_3| \leqslant \frac{1}{x} + \sum_{\lambda_k^* \geqslant x}\frac{h_1}{\lambda_k\lambda_k^*} \leqslant \frac{1}{x} + \frac{1}{x - h_1} \leqslant \frac{3}{x}$$

and

$$|\lambda(x) + \lambda^{**}(x) - \frac{\alpha}{\pi}\log x - A_1 - A_3| \leqslant \frac{13}{x} + \varepsilon(x).$$

Suppose that f is in $\overline{\text{span}}E(\Lambda)$, since $\overline{\text{span}}E(\Lambda) \subset \overline{\text{span}}E(\Lambda \cup \Lambda^{**})$, then from the uniqueness of $E(\Lambda \cup \Lambda^{**})-$expansion $\sum b_n z^{\mu_n}$ of f, where $\Lambda \cup \Lambda^{**} = \{\mu_n : n = 1, 2, \cdots\}$, we see that those coefficients b_n associated with members $\mu_n \in \Lambda^{**}$ distinct from all λ_n are equal to zero. Thus the $E(\Lambda \cup \Lambda^{**})-$expansion reduces to $E(\Lambda)-$expansion whenever $f \in \overline{\text{span}}E(\Lambda)$. Therefore, we can assume, without loss of generality, that there exists a constant A_2 such that (11) holds with $b = \frac{\alpha}{\pi}$. Therefore, the function $g_0(z)$ defined by (12) satisfies (13),(14) and (15). Let

$$\psi_k(z) = \frac{z^2 g_0(z)}{(1+z)^4(z-\lambda_k)}, \quad \psi_k(\lambda_k) = \frac{\lambda_k^2 g_0'(\lambda_k)}{(1+\lambda_k)^4},$$

and

$$h_{k,l}(\zeta) = \int_{L_l} \psi_k(z)\zeta^{-z}dz,$$

where $L_l = \{t\exp\{\frac{\pi}{2}li\} : t \geqslant 0\}(l \in \{-1, 0, 1\})$ are half-lines. As has been shown in (19), then there exists a positive constant A_4 such that

$$|\psi_k(iy)| + |\psi_k'(iy)| + |\psi_k''(iy)|\} \leqslant \frac{A_4 e^{\alpha|y|}}{1+|y|^2} \tag{24}$$

and

$$\limsup_{x\to\infty} x^{-1}\log|\psi_k(x)| = 0 \tag{25}$$

hold for each k. By (15),

$$\limsup_{k\to\infty} \lambda_k^{-1}\log|\psi_k(\lambda_k)| = 0. \tag{26}$$

By (24) and (25), $h_{k,0}(\zeta)$ is analytic in the region $D_0 = \{\zeta : |\zeta| > 1, |\arg\zeta| < \pi\}$, $h_{k,l}(\zeta)(l = \pm1)$ is analytic in the sector $D_l = \{\zeta : \alpha < -l\arg\zeta < \pi\}$ and continuous in the closure $\overline{D_l} = \{\zeta : \alpha \leqslant -l\arg\zeta \leqslant \pi\}$ of D_l. By Cauchy's formula, $h_{k,0}(\zeta)$ can be continued analytically to an analytic function in the region $D_{-1}\bigcup D_0 \bigcup D_1 = \{\zeta = \rho e^{i\phi} : \zeta \notin I_\alpha, |\phi| < \pi\}$, i.e., $h_{k,0}(\rho e^{i\phi}) = h_{k,l}(\rho e^{i\phi})$ for $\rho > 1, \alpha < -l\phi < \pi, l = \pm1$. By (25), $h_0(e^{-\delta}\zeta)$ is bounded in the circular arc $\{\zeta : |\zeta| = 1, |\arg\zeta| < \alpha\}$ for each $\delta > 0$. the linear functionals

$$T_{k,\delta}(\varphi) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \varphi(e^{i\theta})h_{k,0}(e^{-\delta}e^{i\theta})d\theta$$

$$+\frac{1}{2\pi i}\int_0^1 (\varphi(se^{-i\alpha})h_{k,1}(e^{-\delta}se^{-i\alpha}) - \varphi(se^{i\alpha})h_{k,-1}(e^{-\delta}se^{i\alpha})\frac{ds}{s} \quad (\varphi \in H_\alpha)$$

are bounded linear functionals in H_α and satisfy $T_{k,\delta}(\zeta^\lambda) = \psi_k(\lambda)e^{-\delta\lambda}$ for $\lambda \in \mathbb{C}_+$ and $A(\delta) = \sup\{||T_{k,\delta}|| : k = 0, 1, 2, \cdots\} < \infty$. Therefore, $\{e^{\delta\lambda_k}(\psi_k(\lambda_k))^{-1}T_{k,\delta} : k = 1, 2, \cdots\}$ is a biorthogonal system of $E(\Lambda)$. If f belongs to $\overline{\text{span}}E(\Lambda)$, then there exists a sequence of $\Lambda-$polynomials

$$P_l(z) = \sum_{n=1}^{l} a_{n,l}z^{\lambda_n} \in \text{span}E(\Lambda)$$

such that

$$||f - P_l|| \longrightarrow 0 \text{ as } l \longrightarrow \infty.$$

Let

$$\sum_{k=1}^{\infty} a_k z^{\lambda_k} \tag{27}$$

be the $E(\Lambda)-$expansion of f. The biorthogonality of the system

$$\{e^{\delta\lambda_k}(\psi_k(\lambda_k))^{-1}T_{k,\delta} : k = 1, 2, \cdots\}$$

implies that $a_k = e^{\delta\lambda_k}(\psi_k(\lambda_k))^{-1}T_{k,\delta}(f)$ and $a_{k,l} = e^{\delta\lambda_k}(\psi_k(\lambda_k))^{-1}T_{k,\delta}(P_l)$. Therefore

$$|a_k - a_{k,l}| \leqslant ||f - P_l||A(\delta)e^{\delta\lambda_k}|\psi_k(\lambda_k)|^{-1} \quad (k = 1, 2, \cdots)$$

(thus that the sequences $\{a_{k,l}\}$ are independent of δ implies that the sequence $\{a_k\}$ is also independent of δ) and

$$|a_k| \leqslant A(\delta)||f||e^{\delta\lambda_k}|\psi_k(\lambda_k)|^{-1}, \quad k = 1, 2, \cdots .$$

By (26), the series in (27) converges to an analytic function $F(z)$ uniformly on compacts of $\{z : |z| < 1, |\arg z| < \pi\}$. we obtain that, for $z \in \mathrm{int} I_\alpha$, there is $\delta > 0$ such that $|z| < e^{-\delta}$, so

$$\begin{aligned}|f(z) - F(z)| &\leqslant |f(z) - P_l(z)| + |P_l(z) - F(z)|\\ &\leqslant ||f - P_l|| + \sum_{n=1}^{l} |a_{nl} - a_n||z|^{\lambda_n} + \sum_{n=l+1}^{\infty} |a_n||z|^{\lambda_n}.\end{aligned}$$

Letting $l \to \infty$, we obtain that $f(z) = F(z)$ for $z \in \mathrm{int} I_\alpha$. This completes the proof of Theorem.

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