Maximal and minimal forms for generalized Schrödinger operator

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Abstract

Let $0 \leq V \in L^1_{\text{Loc}}(\mathbb{R}^n)$ and $H = (-\Delta)^m + V$ $(m \in \mathbb{N})$ be the generalized Schrödinger type operator. Then there are two a-priori natural nonnegative closed forms associated to the self-adjoint extension of H: the maximal closed form $Q_{\max}$ defined by the sum

$$Q_{\max}(f,f) = Q_0(f,f) + \langle V^{1/2}f, V^{1/2}f\rangle.$$

for any $f \in W^{m,2}(\mathbb{R}^n)$ with $V^{1/2}f \in L^2(\mathbb{R}^n)$ and the minimal closed form $Q_{\min}$ defined by the form closure of $Q_{\max}$ restricted to $C_c^\infty(\mathbb{R}^n)$. If $m = 1$, then it was shown by T. Kato that the maximal and minimal forms are identical, based on his famous positivity inequality. However, for $m \geq 2$, the problem of the consistency seems to have no complete answer in the case of the most general locally integrable potential.

In this paper, the authors prove that $C_c^\infty(\mathbb{R}^n)$ is the form core of the domain $D(Q_{\max})$ for any $0 \leq V \in L^p_{\text{Loc}}((\mathbb{R}^n))$ with some p depending on n, m which greatly improves a form core result of E. B. Davies [5] concerning all smooth non-negative potentials. In particular, we can choose $V \in L^1_{\text{Loc}}((\mathbb{R}^n))$ (the most general locally integrable potential class) if $2m > n$. Finally, the form core result can be applied to establish the sharp bound of the kernel of the semigroup e^{-tH} for $2m > n$.

1 Introduction

Let $m \geq 1,\ n \geq 1$ be any two positive integers. It is well-known that the poly-harmonic operator $H_0 := (-\Delta)^m$ is a nonnegative self-adjoint operator on the Sobolev space $W^{2m,2}(\mathbb{R}^n)$ and associated to a symmetric nonnegative closed form $Q_0(f,g)$ on the form domain $D(\mathcal{Q}_0) = W^{m,2}(\mathbb{R}^n)$:

$$Q_0(f,g) = \langle H_0^{1/2} f, H_0^{1/2} g\rangle = \int_{\mathbb{R}^n} |\xi|^{2m} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} d\xi, \tag{1.1}$$

where $\widehat{f}$ denotes Fourier transform of f and $H_0^{1/2} = (-\Delta)^{m/2}$ is the square root of H_0(e.g. see Kato [12, p.281]), which can be expressed by the Fourier multiplier:

$$(H_0^{1/2} f)^\wedge(\xi) = |\xi|^m \widehat{f}(\xi), \quad f \in W^{m,2}(\mathbb{R}^n). \tag{1.2}$$

Let $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $V \geq 0$. Then there are two a-priori natural nonnegative closed forms associated to the self-adjoint extension of sum $H_0 + V$. Firstly, let

$$D(Q_{\max}) = \left\{ f \in W^{m,2}(\mathbb{R}^n);\ \ V^{1/2} f \in L^2(\mathbb{R}^n) \right\} \tag{1.3}$$

and define the maximal closed form $Q_{\max}$ on $D(Q_{\max})$ by the sum

$$Q_{\max}(f,g) = Q_0(f,g) + \left\langle V^{1/2} f, V^{1/2} g \right\rangle. \tag{1.4}$$

Secondly, note that $C_c^\infty(\mathbb{R}^n) \subset D(Q_{\max})$, then we also can obtain a minimal closed form $Q_{\min}$ by the form closure of $Q_{\max}$ restricted to $C_c^\infty(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n)$. Thus a natural question is that whether the equality $Q_{\max} = Q_{\min}$ holds. Equivalently, we can ask what are the general conditions on V such that the space $C_c^\infty(\mathbb{R}^n)$ is the form core of $Q_{\max}$ with the domain $D(Q_{\max})$?

When $m = 1$ (correspondingly, $-\Delta + V$ is the classical Schrödinger operator), the question was proposed and answered by T. Kato [11] under the most general condition that $0 \leq V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ based on semigroup tools, also see B. Simon [19]. Both authors of [11] and[19] studied the more complex cases with singular magnetic vector potentials. In fact, there was a related and extensively studied question of whether $-\Delta + V$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^n)$ for any $0 \leq V \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$. This question on essential self-adjointness was first conjectured by B. Simon [17], and proved by T. Kato [10] based on his famous distributional inequality:

$$\Delta|\varphi| \geq \Re((\mathrm{sgn}\ \varphi)\ \Delta\varphi), \tag{1.5}$$

for any $\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with $\Delta\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ where $\mathrm{sgn}\ \varphi = \lim_{\epsilon\downarrow 0}\ \varphi/(|\varphi|^2 + \epsilon^2)^{1/2}$. Since then, many studies and methods on the form core problem and essential self-adjointness have been developed (See Simon [22] for more detail reviews and therein references). In particular, Simon [18] found that Kato's inequality (1.5) is actually equivalent to the positivity preserving of semigroup $e^{t\Delta}$ (i.e. $e^{t\Delta}\varphi \geq 0$ for any $L^2 \ni \varphi \geq 0$) or the domination of semigroup (i.e. $|e^{t\Delta}\varphi| \leq e^{t\Delta}|\varphi|$ for any $\varphi \in L^2$), and developed an kind of abstract Kato's inequality version (also see [9], [20]).

However, for $m \geq 2$, since the higher order heat semigroups $e^{-t(-\Delta)^m}$ are not positivity-reserving (see Reed and Simon [14], Davies [6]), so one can't establish the similar Kato inequality as (1.5) (see [18]) or the higher order semigroup domination:

$$(1.6) \qquad |e^{-tH}\varphi| \leq e^{-t(-\Delta)^m}|\varphi|, \quad m \geq 2, \quad \varphi \in L^2(\mathbb{R}^n),$$

where H is the self-adjoint operator associated with the maximal nonnegative closed form $Q_{\max}$ (See (1.4) above). Hence it seems that the study of the C_c^∞-form core question for $Q_{\max}$ is very nontrivial for higher order cases under the most general assumption of $0 \leq V \in L^1_{\text{loc}}$. In particular, the higher order semigroup e^{-tH} loses the connection of stochastic process because of the non-positivity of the kernel, thus it is difficult to use the tools of probability theory based on Feymann-Kac formula for the Schrödinger operator (e.g. see [21]).

On the other hand, it is well known that there is also a famous KLMN-form perturbation method to deal with this kind of problem (see [14]). For instance, if $V \in L^{n/2m}(\mathbb{R}^n)$ for $n > 2m$, then V is the infinitesimally small form perturbation of $(-\Delta)^m$ on $W^{m,2}(\mathbb{R}^n)$ (see Davies and Hinz [7], Zheng and Yao [24]), that is, for any $\epsilon > 0$, there exists a constant $b_\epsilon > 0$ such that

$$(1.7) \qquad |\langle Vf, f\rangle| \leq \epsilon Q_0(f,f) + b_\epsilon \langle f, f\rangle, \quad f \in W^{m,2}(\mathbb{R}^n).$$

Hence $D(Q_{\max}) = W^{m.2}(\mathbb{R}^n)$ and $C_c^\infty(\mathbb{R}^n)$ is the form core of $Q_{\max}$. It is remarkable that the perturbation method can work well for the potential with the sign change and also cover some important singular classes potential, such as Kato class potential including the $O(\frac{1}{|x|^\alpha})$ for some $\alpha > 0$ depending on N, m (e.g. see [7] [16] [24]). However, the global uniformly integrability of V which is necessary to the perturbation also restricts application to many local integrable functions, such as the typical polynomials potentials including square oscillator $|x|^2$ et al. In fact, E. B. Davies [5, p.94] has verified that the $C_c^\infty(\mathbb{R}^n)$ is the form core of the $Q_{\max}$ for any $0 \leq V \in C^\infty(\mathbb{R}^n)$. Clearly, an affirmative answer to general rough potentials would have more interesting applications in the study of spectra and semigroups.

In this note, the authors mainly prove that $C_c^\infty(\mathbb{R}^n)$ is the form core of the domain $D(Q_{\max})$ using only locally integrable condition on V. As a corollary, we also obtain the same result of E. B. Davies concerning all smooth non-negative potentials. Finally, we use the form core result to establish the sharp bound of the kernel of semigroup e^{-tH}.

Our main results are as follows:

Theorem 1.1. *Let* $0 \leq V \in L^p_{\text{loc}}(\mathbb{R}^n)$ *where* $p = n/2m$ *if* $n > 2m$, $p > 1$ *if* $n = 2m$ *and* $p = 1$ *if* $n < 2m$. *Let* $Q_{\max}$ *be the maximal non-negative closed form with the form domain* $D(Q_{\max})$ *defined as in the (1.4). Then the space* $C_c^\infty(\mathbb{R}^n)$ *is a form core of the form* $Q_{\max}$, *that is,* $C_c^\infty(\mathbb{R}^n)$ *is a dense subset of the Hilbert space* $D(Q_{\max})$ *with respect to the norm*

$$(1.8) \qquad \|f\|_{Q_{\max}} := \Big(Q_{\max}(f,f) + (f,f)\Big)^{1/2}$$

Note that, if $0 \leq V \in C(\mathbb{R}^n)$, then $V \in L^\infty_{\text{loc}}(\mathbb{R}^n) \subset L^p_{\text{loc}}(\mathbb{R}^n)$ for any $p \geq 1$. Hence Theorems 1.1 is true for any nonnegative continuous potentials including nonnegative C^∞-class, which cannot be covered by any perturbation class. Thus we immediately obtain the following corollary.

Corallary 1.1. *Let $0 \le V \in C(\mathbb{R}^n)$ and the maximal closed form $Q_{\max}$ be defined by the (1.4). Then the space $C_c^\infty(\mathbb{R}^n)$ is the form core of the $Q_{\max}$.*

Remark 1.1. (i) When $n < 2m$, the local integrable conditions of V in Theorems 1.1 are optimal. If $n \ge 2m \ge 4$, then the minimal integrable indexes p of the positive potential V are unknown at present as concerned as the form core. Moreover, the poly-harmonic operator $(-\Delta)^m$ studied in this paper can be replaced with some trivial changes by any positive elliptic operator $P(D)$ of order $2m$.

(ii) The potential V in Theorem 1.1 also can be allowed to contain a small negative part. Let $V = V_+ - V_-$. If the positive part V_+ is the potential of Theorem 1.1 and the negative part V_- satisfies the form perturbation, i.e. there exist positive constants $a < 1$ and $b > 0$ such that

$$(1.9)\quad \int_{\mathbb{R}^n} V_-|f|^2 dx \le a\Big(\int_{\mathbb{R}^n}((-\Delta)^m f)\overline{f}dx + \int_{\mathbb{R}^n} V_+|f|^2dx\Big) + b\int_{\mathbb{R}^n}|f|^2dx, \quad f \in C_c^\infty(\mathbb{R}^n).$$

then the form sum $\int |(-\Delta)^{m/2} f|^2 dx + \int V|f|^2dx$ is a semi-bounded closed form on $W^{m,2}(\mathbb{R}^n) \cap \mathfrak{D}(V_+^{1/2})$ and $C_c^\infty(\mathbb{R}^n)$ is the form core. The characterizations of V satisfying the estimate (1.9) is well known and can be found in Mazya [13, Chapter 12]. In particular, the inequality (1.9) holds if V belongs to Kato class $K_{2m}(\mathbb{R}^n)$, which was introduced by Kato[10] for $m = 1$ and generalized to higher order cases in Davies and Hinz [7], Zheng and Yao [24], also see Schechter [16] for some similar classes adapted to general partial differential operators.

(iii) Given any domain $\Omega \subset \mathbb{R}^n$, the maximal and minimal forms are dramatically different even in the case of $V = 0$. For instance, as $m = 1$, there exist two clearly different forms associated with two different self-adjoint Laplaces: $-\Delta_D$ with Dirichlet boundary condition and $-\Delta_N$ with Newmann boundary condition, and their form domains are the Sobolev spaces $W_0^{m,2}(\Omega)$ and $W^{m,2}(\Omega)$, respectively. However, let Q_0^Ω be the closure of the form $(f, g) \to \int_\Omega (-\Delta)^m f\overline{g}dx$ on $C_c^\infty(\Omega) \times C_c^\infty(\Omega)$, and $0 \le V \in L^p_{\rm loc}(\Omega)$ with the same p as in Theorem 1.1. Then similarly, we can show that the form sum

$$(1.10)\qquad Q^\Omega(f,g) = Q_0^\Omega(f,g) + \langle V^{1/2}f, V^{1/2}g\rangle$$

is closed on $D(Q^\Omega) = \{W_0^{m,2}(\Omega);\ V^{1/2}f \in L^2(\Omega)\}$ and $C_c^\infty(\Omega)$ also is its form core.

The paper is organized as follows: Section 2 mainly is devoted to the proof of theorems and in Section 3 we give an application to the sharp bound of the kernel of $e^{-t((-\Delta)^m+V)}$ for $m \ge 2$ under the same assumption on V as in Theorem 1.1.

2 The proofs of results

In this section, we will prove Theorems 1.1. Our main method can be viewed as the continuation of the semigroup approach used by Kato [11] and Simon [19] for the second-order cases. Comparing with the Schrödinger operators (i.e. $m = 1$), since $e^{-t(-\Delta)^m}$ $(m \ge 2)$ are not positivity reserving for any $t > 0$, it seems hard to prove the higher order Schrödinger semigroup e^{-tH} has the "ultracontractive" regularity from L^2 to L^∞ for $t > 0$ by the Trotter-Kato product formula (see e.g. Davies [3]). Nevertheless, the analyticity of e^{-tH} and Sobolev embedding can help to gain partial L^p-L^q bounds again. Now let us begin with the following useful lemma.

Lemma 2.1. *Let $f \in W^{m,2}(\mathbb{R}^n)$ and $\theta = \frac{n}{2m}(1 - \frac{2}{q})$ where $2 \le q \le 2n/(n-2m)$ if $n > 2m$, $2 \le q < \infty$ if $n = 2m$ and $2 \le q \le \infty$ if $n < 2m$. Then $f \in L^q(\mathbb{R}^n)$ and there exists a constant $C_{m,n} > 0$ such that the following inequality holds:*

$$\|f\|_{L^q(\mathbb{R}^n)} \le C_{m,n} \|(-\Delta)^{m/2} f\|^{\theta}_{L^2(\mathbb{R}^n)} \|f\|^{1-\theta}_{L^2(\mathbb{R}^n)} \tag{2.1}$$

where $(-\Delta)^{m/2}$ is understood as the square root of $(-\Delta)^m$ on $L^2(\mathbb{R}^n)$ (also see the (1.2) of Section 1).

It is obvious that $f \in L^q(\mathbb{R}^n)$ is a corollary of the classical Sobolev embedding Theorem $W^{m,2}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$. For the embedding inequality (2.1), it seems to be very basic and also can be simply proved by the methods of Fourier analysis. For convenience, its proof is given as follows:

Proof. We begin with the proof for the case $n > 2m$. By the Hardy-Littlewood-Sobolev inequality (see [23, p. 354]), there exists some constant $c_{m,n} > 0$ such that

$$\|f\|_{L^{\frac{2n}{n-2m}}} \le c_{m,n} \|(-\Delta)^{m/2} f\|_{L^2}, \qquad f \in W^{m,2}(\mathbb{R}^n).$$

Thus $\forall q \in [2, \frac{2n}{n-2m}]$, it follows from a simple interpolation argument that

$$\|f\|_{L^q} \le \|f\|^{\theta}_{L^{\frac{2n}{n-2m}}} \|f\|^{1-\theta}_{L^2} \le C_{m,n} \|(-\Delta)^{m/2} f\|^{\theta}_{L^2} \|f\|^{1-\theta}_{L^2}$$

for some constant $C_{m,n} > 0$ and $\theta = \frac{n}{2m}(1 - \frac{2}{q})$.

Next, let us turn to the proof of the case $n \le 2m$. Let q satisfy the assumption of Lemma 2.1 and q' be the conjugate index of q. If we can prove that $\widehat{f} \in L^{q'}(\mathbb{R}^n)$ for $n \le 2m$ and

$$\|\widehat{f}\|_{L^{q'}(\mathbb{R}^n)} \le C_{m,n} \|(-\Delta)^{m/2} f\|^{\theta}_{L^2(\mathbb{R}^n)} \|f\|^{1-\theta}_{L^2(\mathbb{R}^n)}, \tag{2.2}$$

for some constants $C_{m,n} > 0$ and $\theta = \frac{n}{2m}(1 - \frac{2}{q})$, then Lemma 2.1 can be concluded by using Young's inequality.

Finally, we come to prove the inequality (2.2) with $q' > 1$ if $n = 2m$ and $q' \ge 1$ if $n < 2m$. Consider

$$\begin{aligned} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^{q'} d\xi \le & \int_{B(0,R)} |\widehat{f}(\xi)|^{q'} d\xi + \int_{\mathbb{R}^n \setminus B(0,R)} ||\xi|^m \widehat{f}(\xi)|^{q'} |\xi|^{-mq'} d\xi \\ \le & C R^{n(1-\frac{q'}{2})} \|f\|^{q'}_{L^2} + C R^{(n - \frac{2q'm}{2-q'})(1-\frac{q'}{2})} \|(-\triangle)^{m/2} f\|^{q'}_{L^2}, \end{aligned} \tag{2.3}$$

where $B(0, R)$ is the ball centered at origin with radius R which will be chosen later. In fact, we use the Hölder's inequality and the fact that $n < \frac{2q'm}{2-q'}$ in the last inequality of (2.3), then the desired (2.2) follows by choosing

$$R = \left(\|f\|_{L^2} \|(-\triangle)^{m/2} f\|_{L^2}^{-1} \right)^{-\frac{1}{m}},$$

in the last inequality of (2.3). Thus we finish the proof of lemma. □

The proof of Theorem 1.1:

Firstly, we will prove that the subset $L^q(\mathbb{R}^n) \cap D(Q_{\max})$ is a form core of $Q_{\max}$, where q is any index defined in Lemma 2.1. To this aim, let H be the non-negative self-adjoint operator with the domain $\mathfrak{D}(H)$ associated to the form $Q_{\max}$ defined in the (1.4), see Kato [12]. Then it is known that the domain $\mathfrak{D}(H)$ is a form core of $Q_{\max}$ and H generates an analytic semigroup e^{-tH} on $L^2(\mathbb{R}^n)$ which satisfy that

$$\|e^{-tH}\|_{L^2-L^2} + \|tHe^{-tH}\|_{L^2-L^2} \le C \tag{2.4}$$

for all $t > 0$. Let $f_t := e^{-tH} f$ for $f \in L^2(\mathbb{R}^n)$ and $t > 0$. Since $\mathfrak{D}(H) \subset D(Q_{\max}) \subset W^{m,2}(\mathbb{R}^n)$ thus we have that $f_t \in W^{m,2}(\mathbb{R}^n)$. Hence from Lemma 2.1 we have

$$\|f_t\|_{L^q(\mathbb{R}^n)} \le C_{m,n} \|(-\Delta)^{m/2} f_t\|_{L^2(\mathbb{R}^n)}^{\theta} \|f_t\|_{L^2(\mathbb{R}^n)}^{1-\theta}. \tag{2.5}$$

where $\theta = \frac{n}{2m}(1 - \frac{2}{q}) \in [0, 1]$ and q is any index defined in Lemma 2.1. On the other hand, by the (1.4) of Section 1 and Cauchy-Schwarz inequality

$$\begin{aligned} \|(-\Delta)^{m/2} f_t\|_{L^2(\mathbb{R}^n)} &\le \big(Q_{\max}(f_t, f_t)\big)^{1/2} = \big(\langle Hf_t, f_t\rangle\big)^{1/2} \\ &\le \|Hf_t\|_{L^2(\mathbb{R}^n)}^{1/2} \|f_t\|_{L^2(\mathbb{R}^n)}^{1/2}. \end{aligned} \tag{2.6}$$

Therefore by combining (2.5) with (2.6), it follows from (2.4) that

$$\begin{aligned} \|e^{-tH}\|_{L^q(\mathbb{R}^n)} = \|f_t\|_{L^q(\mathbb{R}^n)} &\le C \|Hf_t\|_{L^2(\mathbb{R}^n)}^{\theta/2} \|f_t\|_{L^2(\mathbb{R}^n)}^{1-\theta/2} \\ &\le C' t^{-\theta/2} \|f\|_{L^2(\mathbb{R}^n)} = C' t^{-\frac{n}{2m}(\frac{1}{2}-\frac{1}{q})} \|f\|_{L^2(\mathbb{R}^n)}, \end{aligned}$$

which states that e^{-tH} is bounded from $L^2(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for any $t > 0$. This implies that

$$\mathrm{Ran}(e^{-H}) \subset L^q(\mathbb{R}^n). \tag{2.7}$$

Now, if we can show that $\mathrm{Ran}(e^{-H})$ is a form core of $Q_{\max}$, then the embedding (2.7) will conclude that $L^q(\mathbb{R}^n) \cap D(Q_{\max})$ is a form core of $Q_{\max}$. In fact, since $\mathfrak{D}(H)$ is a form core of $Q_{\max}$, hence it suffices to prove that $\mathrm{Ran}(e^{-H})$ is dense in $\mathfrak{D}(H)$ in the sense of the norm (1.8). Note that $\mathrm{Ran}(e^{-H}) \subset \mathfrak{D}(H)$ is an invariant subspace of semigroup e^{-tH} (i.e. a subspace W is invariant if $e^{-tH}(W) \subseteq W$ for any $t > 0$), thus if we can prove that $\mathrm{Ran}(e^{-H})$ is also dense in $L^2(\mathbb{R}^n)$, then it follows from semigroup property (see e.g. Reed and Simon [14]) that $\mathrm{Ran}(e^{-H})$ will be the operator core of H, that is, $\mathrm{Ran}(e^{-H})$ is dense in $\mathfrak{D}(H)$ in the sense of the graph norm

$$\|f\|_H := \big(\|Hf\|_{L^2}^2 + \|f\|_{L^2}^2\big)^{1/2} \tag{2.8}$$

which is stronger than the form norm (1.8). Therefore, finally, we just need to prove that $\mathrm{Ran}(e^{-H})$ is dense in $L^2(\mathbb{R}^n)$, which equivalently, say that $\mathrm{Ker}(e^{-H})=\{0\}$ in view of the self-adjointness of the e^{-H}. Now let $e^{-H}f=0$ for some $f\in L^2(\mathbb{R}^n)$. Then we have $e^{-tH}f=0$ for $t\geq 1$. Since the semigroup $e^{-tH}f$ is analytic on $t>0$, so $e^{-tH}f=0$ for any $t>0$, which deduces $f=0$ as $t\downarrow 0$. Thus we can conclude that the subset $L^q(\mathbb{R}^n)\cap D(Q_{\max})$ is a form core of $Q_{\max}$.

Next, we will show that the set

$$\mathcal{A}_q := L^q_{\mathrm{comp}}(\mathbb{R}^n)\cap D(Q_{\max})$$

is a form core of $Q_{\max}$ for any q defined in Lemma 2.1, where $L^p_{\mathrm{comp}}(\mathbb{R}^n)$ is the subset of $L^p(\mathbb{R}^n)$ with compact support. To this end, let $\psi\in C_c^\infty(\mathbb{R}^n)$ and $f\in D(Q_{\max})=W^{m,2}(\mathbb{R}^n)\cap\mathfrak{D}(V^{1/2})$, where $\mathfrak{D}(V^{1/2})=\{f\in L^2;\quad V^{1/2}f\in L^2\}$. Then clearly, $\psi f\in\mathfrak{D}(V^{1/2})$. Moreover, by the Leibnitz' formula we have

$$D^\alpha(\psi f)=\sum_\gamma C_{m,\gamma}D^\gamma f D^{\alpha-\gamma}\psi \tag{2.9}$$

for any $|\alpha|\leq m$. Hence it follows from (2.9) that $\psi f\in W^{m,2}(\mathbb{R}^n)$ and thus $\psi f\in D(Q_{\max})$. Let $\eta\in C_c^\infty(\mathbb{R}^n)$, $0\leq\eta\leq 1$, $\mathrm{supp}\,\eta\subset B(0,2)$ and $\eta(x)=1$ for $x\in B(0,1)$, where $B(x,r)$ is the ball centered at x with radius r. For any $f\in L^q(\mathbb{R}^n)\cap D(Q_{\max})$, set $f_k=\eta(\cdot/k)f$ for $k=1,2\cdots$, then $f_k\in\mathcal{A}_q$ and there exists some constant $C>0$ independent of k such that

$$\|f_k-f\|_{Q_{\max}}\leq C\big(\|f_k-f\|_{W^{m,2}}+\|V^{1/2}(f_k-f)\|_{L^2}\big) \tag{2.10}$$

Hence by the (2.9), (2.10) and the domination convergence theorem it is easy to get that $f_n\to f$ as $n\to\infty$ in the sense of the norm (1.8), which shows that the set $\mathcal{A}_q$ is dense subset of $L^q(\mathbb{R}^n)\cap D(Q_{\max})$. Thus the set $\mathcal{A}_q$ is also a form core of $Q_{\max}$ for any q defined in Lemma 2.1.

Finally, in order to prove that $C_c^\infty(\mathbb{R}^n)$ is a core of the form $Q_{\max}$, it suffices to show that $C_c^\infty(\mathbb{R}^n)$ is dense in the set $\mathcal{A}_q$ (i.e. $L^q_{\mathrm{comp}}(\mathbb{R}^n)\cap D(Q_{\max})$) for some q from Lemma 2.1. Let $0\leq\varphi\in C_c^\infty(\mathbb{R}^n)$ satisfy $\int\varphi(x)dx=1$ and $\mathrm{supp}\,\varphi\subset B(0,1)$. For any $f\in\mathcal{A}_q$, set $f_\delta=\varphi_\delta * f$ where $\varphi_\delta(x)=\delta^{-n}\varphi(x/\delta)$ for any $\delta>0$. Then obviously, $f_\delta\in C_c^\infty(\mathbb{R}^n)$. Since $f\in D(Q_{\max})=W^{m,2}(\mathbb{R}^n)\cap\mathfrak{D}(V^{1/2})$, we have

$$D^\alpha f_\delta\to D^\alpha f \quad \text{in } L^2(\mathbb{R}^n)$$

as $\delta\to 0$ for all $0\leq|\alpha|\leq m$. Hence $\|f_\delta-f\|_{W^{m,2}}\to 0$ as $\delta\to 0$. In view of the (2.10), it suffices to prove $\|V^{1/2}(f_\delta-f)\|_{L^2}\to 0$ as $\delta\to 0$. Note that f_δ and f both have compact supports contained in some bounded ball K for $\delta\leq 1$. If $n\geq 2m$, then by Hölder's inequality and the assumption $V\in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ we have that

$$\int_K V|f_\delta-f|^2dx\leq\Big(\int_K V^pdx\Big)^{1/p}\Big(\int_{\mathbb{R}^n}|f_\delta-f|^qdx\Big)^{2/q}\to 0, \tag{2.11}$$

as $\delta\to 0$, where $p=n/2m$ and $q=2n/(n-2m)$ if $n>2m$ (we choose $q=2p/(p-1)$ for some $p>1$ if $n=2m$). Thus we have shown that $C_c^\infty(\mathbb{R}^n)$ is a core of the form $Q_{\max}$ if

$n \geq 2m$. If $n < 2m$, then we can choose $q = \infty$ and $f \in \mathcal{A}_\infty = L^\infty_{\text{comp}}(\mathbb{R}^N) \cap D(Q_{\max})$. Since $|f_\delta| \leq |f| \in L^\infty_{\text{comp}}$ and $\|f_\delta - f\|_{L^2} \to 0$ as $\delta \to 0$, we can choose some subsequence $\delta_i \to 0$ such that $V(f_{\delta_i} - f) \to 0$ a.e. as $\delta_i \to 0$, and $|V(f_{\delta_i} - f)| \leq C|V| \in L^1(K)$. Thus it follows from Lebesgue domination theorem that $\|V^{1/2}(f_{\delta_i} - f)\|_{L^2} \to 0$ as $\delta_i \to 0$. Hence we have completed the whole proof of Theorem 1.1.

As a corollary of Theorem 1.1, we will give a specific description of the domain $\mathfrak{D}(H)$ of the self-adjoint operator H associated to the maximal form $Q_{\max}$.

Proposition 2.1. *Let V satisfy the same condition as Theorem 1.1 and H be the unique self-adjoint operator associated with the form $Q_{\max}$ defined by the (1.4). Then*

$$\mathfrak{D}(H) = \left\{ f \in D(Q_{\max});\ \left((-\Delta)^m f + Vf\right)_{dist} \in L^2(\mathbb{R}^n) \right\} \tag{2.12}$$

and $Hf = \left((-\Delta)^m f + Vf\right)_{dist}$ in the distributional sum sense.

Proof. Let $\widetilde{H}$ be an operator given by $\widetilde{H}f = (-\Delta)^m f + Vf$ in distributional sum sense with the domain

$$\mathfrak{D}(\widetilde{H}) = \left\{ f \in D(Q_{\max}); \left((-\Delta)^m f + Vf\right)_{dist} \in L^2(\mathbb{R}^n) \right\} \tag{2.13}$$

Note that as $f \in Q_{\max}$ and $V \in L^1_{\text{loc}}(\mathbb{R}^n)$, then $V^{1/2} \in L^2_{\text{loc}}(\mathbb{R}^n)$ and $V^{1/2}f \in L^2_{\text{loc}}(\mathbb{R}^n)$, which gives $Vf \in L^1_{\text{loc}}(\mathbb{R}^n)$ by Hölder inequality. This explains the meaning of distributional sum in the definition of $\widetilde{H}$ above. In the sequel, we will prove that $\widetilde{H} = H$.

First, by the construction of H, we have

$$\mathfrak{D}(H) = \left\{ f \in D(Q_{\max}); \exists g \in L^2 \text{ such that } Q_{\max}(f, u) = \langle g, u \rangle;\ \forall u \in D(Q_{\max}) \right\}. \tag{2.14}$$

Let $f \in \mathfrak{D}(H)$. Then since $C_c^\infty(\mathbb{R}^n) \subset D(Q_{\max})$, by the definition of distributional derivative and the fact that $Vf \in L^1_{\text{loc}}(\mathbb{R}^n)$, we get that for any $u \in C_c^\infty(\mathbb{R}^n)$

$$Q_{\max}(f, u) = \int_{\mathbb{R}^n} \widetilde{H}f(x)u(x)dx = \langle g, u \rangle \tag{2.15}$$

Hence $f \in \mathfrak{D}(\widetilde{H})$ and $\widetilde{H}f = g$, i.e. $H \subseteq \widetilde{H}$. On the other hand, let $f \in \mathfrak{D}(\widetilde{H})$ and $g = \widetilde{H}f \in L^2$. Then by (2.15) again we can get that $Q_{\max}(f, u) = \langle g, u \rangle$ for any $u \in C_c^\infty(\mathbb{R}^n)$. Since $C_c^\infty(\mathbb{R}^n)$ is the form core of $Q_{\max}$ by Theorem 1.1, hence it follows from the density arguments and the (2.14) that $f \in \mathfrak{D}(H)$ and $Hf = g$, i.e. $\widetilde{H} \subseteq H$. Thus we complete the proof of Proposition 2.1. □

3 An application to sharp bounds of semigroup kernel

Let $H = H_0 + V$ be the self-adjoint operator associated with the maximal form $Q_{\max}$ defined by the (1.4), where H_0 is any nonnegative homogeneous elliptic operator of order $2m$ and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Since $H \geq 0$, it generates a contractive semigroup e^{-tH} on $L^2(\mathbb{R}^n)$. If $m = 1$, it is known from the theory of the Dirichlet forms that Schrödinger semigroup can be extended to a contractive semigroup $L^p(\mathbb{R}^n)$ for any $1 \leq p < \infty$ (see [3, Theorem 1.3.5]). The integral expression of e^{-tH} is given by the famous Feynman-Kac formula, and the whole theory was profoundly enriched by the connection to the Brownian motion (see [21], [22]). In particular, if $H = -\Delta + V$, then the kernel $K(t, x, y)$ of Schrödinger semigroup e^{-tH} satisfies the following sharp estimates:

$$0 \leq K(t, x, y) \leq e^{t\Delta}(x, y) = (4\pi t)^{-n/2} \exp\Big\{ -\frac{|x-y|^2}{4t} \Big\}. \tag{3.1}$$

However, for $2m \geq 4$, the situations are much complicated and depending upon dimension n and order m. Generally, if $n > 2m \geq 4$, then the semigroup e^{-tH} has the L^p-extension for $p \in [p_c, p_c']$ where $p_c = 2n/(n+2m)$ and for any value of p outside the interval $[p_c, p_c']$ there exist some counter-examples constructions of H such that e^{-tH} are unbounded on $L^p(\mathbb{R}^n)$ for any $t > 0$ as $n \geq 2m + 3$ (see [5]). However, if $n < 2m$, then e^{-tH} can be extended into a strongly continuous semigroup on $L^p(\mathbb{R}^n)$ for any $p \in [1, \infty)$ and its kernel has the following upper bound with some parameter $c, d > 0$(see Barbatis and Davies [1, Proposition 5.2]):

$$|K(t, x, y)| \leq Ct^{-n/2m} \exp\Big\{ -c\frac{|x-y|^{2m/2m-1}}{t^{1/2m-1}} + dt\Big\}. \tag{3.2}$$

In particular, if $H = (-\Delta)^m$, then the estimate (3.2) can be improved as follows:

$$|K(t, x, y)| \leq C_r t^{-\frac{n}{2m}} \exp\Big\{ -d_m\frac{|x-y|^{2m/2m-1}}{rt^{1/2m-1}} \Big\}, \tag{3.3}$$

where $C_r > 0$ for all $r > 1$ and

$$d_m = (2m-1)(2m)^{-\frac{2m}{2m-1}} \sin\frac{\pi}{4m-2}.$$

As an application of Theorem 1.1, we can easily extend the sharp bound (3.3) to the case $H = (-\Delta)^m + V$ with any nonnegative potential V as following:

Theorem 3.1. *Let $H = (-\Delta)^m + V$ be the self-adjoint operator associated with the maximal form $\mathcal{Q}_{\max}$ defined by the (1.4). If $2m > n$ and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$, then*

$$|K(t, x, y)| \leq C_r t^{-\frac{n}{2m}} \exp\Big\{ -d_m\frac{|x-y|^{2m/2m-1}}{rt^{1/2m-1}} \Big\}, \tag{3.4}$$

where $C_r > 0$ for all $r > 1$ and

$$d_m = (2m-1)(2m)^{-\frac{2m}{2m-1}} \sin\frac{\pi}{4m-2}.$$

Proof. By Theorem 1.1, we know that $C_c(\mathbb{R}^n)$ is the form core of $\mathcal{Q}_{\max}$. Thus the desired sharp bound can immediately be obtained by combining the methods used in Theorem 4.3 and Proposition 5.2 of Barbatis and Davies [1]. □

Remark 3.1. When H_0 is any nonnegative homogeneous elliptic operator of order $2m$ with $2m > n$ and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$, the sharp kernel bound above is also true for the general operator $H_0 + V$ only with some different constant d_m. Moreover, such methods and results above can also be extended to the signed potential V satisfying the following strongly subcritical condition that there exists some constant $\mu \in (0, 1)$ such that

$$\int_{\mathbb{R}^n} V_- |f|^2 dx \leq \mu\Big(\int_{\mathbb{R}^n} |H_0^{1/2} f|^2 dx + \int_{\mathbb{R}^n} V_+ |f|^2 dx\Big), \tag{3.5}$$

holds for all $f \in W^{m,2}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} V_+ |f|^2 dx < \infty$ (see e.g. [8]).

References

[1] G. Barbatis and E. B. Davies, *Sharp bounds on heat kernels of higher order uniformly elliptic operators*, J. Oper. Theory. **36** (1996), 179-198.

[2] E. B. Davies, *One Paramenter semigroups,* Academic Press, San Diego, 1980.

[3] E. B. Davies, *Heat kernel and spectral theory,* Cambridge Univ. Press, 1989.

[4] E. Davies, *Uniformly elliptic operators with mesrsurable cosfficients,* J. Funct. Anal. **132** (1995), 141-169.

[5] E. B. Davies, *Limits on L^p regularity of self-adjoint elliptic operators,* J. Diff. Equa. **135** (1997), 83-102.

[6] E. B. Davies, *Long time asympototics of the fourth order parabolic equations,* J. Anal. Math. **67** (1995), 323-345.

[7] E. B. Davies and A. M. Hinz, *Kato class potentials for higher-order elliptic operators*, J. London Math. Soc., **58** (1998), 669-678.

[8] Q. Deng, Y. Ding and X. Yao, *Riesz transforms associated to higher order Schrödinger operators,* Preprint.

[9] H. Hess, R. Schrader and D. A. Uhlenbrock, *Domination of semigroup and generalization of Kato's inequality,* Duke Math. J., **44** (1977), 893-904.

[10] T. Kato, *Schrödinger operators with singular potentials*, Isreal J. Math., **13** (1973), 135-148.

[11] T. Kato, *Remarks on schrödinger operators with vector potentials*, Integral Equations and Operator Theory, **1** (1978), 103-113.

[12] T. Kato, *Perturbation Theory for Linear Operators*, 2nd edtion, Springer-Verlag, 1980.

[13] V. G. Maz'ya, *Sobolev spaces, with the applications to elliptic partial differential equations,* Second edition, Springer, 2011.

[14] M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness,* Academic Press, New York, 1975.

[15] M. Reed and B. Simon, *Methods of modern mathematical physics. IV. Analysis of operators,* Academic Press, New York, 1978.

[16] M. Schechter, *Spectra of partial differential operators,* 2nd edn Elsevier Science Publishers B.V., Amsterdam, 1986.

[17] B. Simon, *Essential self-adjointness of schrödinger operators with positive potentials,* Math. Ann., **201** (1973), 211-220.

[18] B. Simon, *An abstract Kato's inequality for generators of positivity preserving semigroups,* Indiana Univ. Math. J., **26** (1977), 1067-1073.

[19] B. Simon, *Maximal and minimal schrödinger forms*, J. Operator. Theory. **1** (1979), 37-47.

[20] B. Simon, *Kato's inequality and the comparison of semigroups,* J. Funct. Anal., **32** (1979), 97-101.

[21] B. Simon, *Functional integration and quantum physics* Academic Press, New York, 1979.

[22] B. Simon, *Schrodinger semigroups*, Bull. Amer. Math. Soc. N.S. **7** (1982), 447-526.

[23] E. M. Stein, *Harmonic Analysis, Real Variable Methods, Orthogonality, and Oscillatory Integrals,* Princeton Univ. Press, Princeton, NJ, 1993.

[24] Q. Zheng and X. Yao, *Higher-order Kato class potential for Schrödinger operators*, Bull. London Math. Soc. **41** (2009), 293-301.